I. Simplifying

1. Suppose that \(a, b\) are nonzero real numbers and \(|ab| \neq 1\). For integers \(x, y\) simplify

\[
\frac{(a+b^{-1})^x(a-b^{-1})^y}{(b+a^{-1})^x(b-a^{-1})^y}.
\]

**Solution.**

\[
\frac{(a+b^{-1})^x(a-b^{-1})^y}{(b+a^{-1})^x(b-a^{-1})^y} = \left(\frac{a}{b}\right)^{x+y}.
\]

2. Verify that \(\sqrt{2} \sqrt{2} = \sqrt{2^2} = 2\).

**Solution.** By definition \(\sqrt{a} = a^{1/p}\) for all \(a \geq 0, p > 0\). Hence

\[
\sqrt{2} \sqrt{2} = 2^{1/2} \sqrt{2} = 2 \sqrt{2} = (2^{1/2})^{2^{1/2}}.
\]

3. Let \(x \geq 0\) and let \(k, l, m, n\) be positive integers. Simplify the following expressions:

(a) \(\sqrt[k]{x} \sqrt[m]{x} - \sqrt[kn+lm]{x}
\]

(b) \(\sqrt[k]{x^k} \cdot \sqrt[n]{x^{lm-kn}}
\]

(c) \(\sqrt[n]{(\sqrt[k]{x})^m} - \sqrt[n]{x^{km}}
\]

**Solution.** 0, 1, 0.

4. Let \(a, b, c, d \in \mathbb{R}\). Simplify the following expressions:

(a) \((a^2 + b^2)(c^2 + d^2) - (ac - bd)^2 - (ad + bc)^2
\]

(b) \((a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 - (ad - bc)^2
\]

(c) \(\frac{1}{2} (a-b)((a^2 + b^2 + (a + b)^2)
\]

**Solution.** 0, 0, \(a^3 - b^3\).

II. Proving identities

1. Show that for all real \(a, b\) : \((a^2 + b^2 + (a + b)^2)^2 = 2(a^4 + b^4 + (a + b)^4)\).

Deduce Candido’s identity for the Fibonacci numbers \(F_n\) defined for any \(n \in \mathbb{N}\) by \(F_n = F_{n+1} + F_n\) (with initial values \(F_0 = 0, F_1 = 1\)):

\[
(F_n + F_{n+1} + F_{n+2})^2 = 2(F_n^4 + F_{n+1}^4 + F_{n+2}^4).
\]

See also: [https://en.wikipedia.org/wiki/Giacomo_Candido#Candido.27s_identity](https://en.wikipedia.org/wiki/Giacomo_Candido#Candido.27s_identity)
Solution. Expand explicitly all products or apply the binomial and multinomial identities.

2. For the Fibonacci numbers $F_n$ defined for any $n \in \mathbb{N}$ by $F_{n+2} = F_{n+1} + F_n$ with initial values $F_0 = 0$, $F_1 = 1$ show Cassini’s formula

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}. \quad (2)$$

Formulate Cassini’s identity for general initial values $F_0 = a_0$, $F_1 = a_1$.

For further extensions see: https://en.wikipedia.org/wiki/Cassini_and_Catalan_identities

Solution. By induction show that

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n}(F_0 F_2 - F_1^2) \quad (3)$$

where in the special case $F_0 F_2 - F_1^2 = -1$ and in the general case $F_0 F_2 - F_1^2 = a_0^2 - b_0^2 + a_0 b_0$. Indeed, $(3)$ trivially holds for $n = 0$ and if it holds for a given $n$ then using the recurrence relations $F_n = F_{n+2} - F_{n+1}$ and $F_{n+1} = F_{n+3} - F_{n+2}$:

$$( -1)^n (F_0 F_2 - F_1^2) = F_n F_{n+2} - F_{n+1}^2$$

$$= (F_{n+2} - F_{n+1}) F_{n+2} - F_{n+1} (F_{n+3} - F_{n+2})$$

$$= F_{n+2}^2 - F_{n+1} (F_{n+3} - F_{n+2}) = - (F_{n+1} (F_{n+3} - F_{n+2})$$

proving the claim.

3. For any real $x$ let $\lfloor x \rfloor$ denote the integer part of $x$ (also called the floor function), that is $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$. Show that for all $x$:

$$\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor, \quad \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = \lfloor 3x \rfloor.$$

Solution. Let $\lfloor x \rfloor = n \geq 0$, that is $x \in [n, n+1]$. If $x \in [n, n + \frac{1}{2}]$ then also $\lfloor x + \frac{1}{2} \rfloor = n$ and since $2x \in [2n, 2n + 1]$ it follows $\lfloor 2x \rfloor = 2n$ which proves the identity in this subcase. If $x \in [n + \frac{1}{2}, n + 1]$ then $\lfloor x + \frac{1}{2} \rfloor = n + 1$ and since $2x \in [2n + 1, 2n + 2]$ it follows $\lfloor 2x \rfloor = 2n + 1$ proving the identity whenever $\lfloor x \rfloor = n \geq 0$. To prove the second identity you have to distinguish the three cases $x \in [n, n + \frac{1}{2}]$, $x \in [n + \frac{1}{2}, n + \frac{3}{2}]$, and $x \in [n + \frac{3}{2}, n + 1]$. 

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III. Polynomial Division and Factorization

1. Determine $a$ such that $x + 1$ divides $x^3 + 5x^2 + ax + 1$.

**Solution.** $a = 5$.

2. Determine $a, b$ such that $x + 1$ and $x + 2$ divide $x^3 + ax^2 + bx + a - b$.

**Solution.** $a = \frac{13}{4}$, $b = \frac{11}{4}$ which are the solutions of the linear system

$$2a - 2b - 1 = 0, \quad 5a - 3b - 8 = 0.$$

3. Factorize the following expressions:

   (a) $x^4 + 1$ 
   (b) $x^6 + 1 - (x^2 + 1)^3$ and deduce a factorization of $x^6 + 1$.

**Solution.**

(a) $x^4 + 1 = (x^2 + 1)^2 - 2x^2 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$.

(b) $x^6 + 1 - (x^2 + 1)^3 = -3x^2(x^2 + 1)$ and therefore

$$x^6 + 1 = (x^2 + 1)((x^2 + 1)^2 - 3x^2) = (x^2 + 1)(x^2 + \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1).$$

4. Let $a, b, c, d, \lambda \in \mathbb{R}$ factorize the following expression: $(ac - \lambda bd)^2 + (ad + \lambda bc)^2$.

**Solution.**

$$(ac - \lambda bd)^2 + (ad + \lambda bc)^2 = (a^2 + \lambda b^2)(c^2 + \lambda d^2).$$

5. Determine $a$ such that $x^2 + 2$ divides $x^4 + 2x^3 + ax^2 + 4x + 2$.

**Solution.** $a = 3$.

6. Determine $a, b \in \mathbb{R}$ such that $x^2 + x + 1$ divides $x^6 + x^5 + ax^4 + x^3 + 2x^2 + bx + 1$.

**Solution.** $a \in \mathbb{R}$ and $b = 3 - a$.

7. Find $A, B$ such that

$$\frac{13x(x - 2)}{(x^2 + 1)(9x^2 - 4)} = \frac{2x + 1}{x^2 + 1} + \frac{A}{3x + 2} + \frac{B}{3x - 2}$$

**Solution.** $A = -4$, $B = -2$.  
