Squares, Square Roots and the Absolute Value

I. Squares, Square Roots

1. Suppose \( a \geq b \geq 0 \) and \( c \geq d \geq 0 \). Prove that
\[
\sqrt{a^2 - b^2} \sqrt{c^2 - d^2} \leq ac - bd.
\]

Solution. Since \( a^2 \geq b^2 \geq 0 \) and \( c^2 \geq d^2 \geq 0 \) the inequality follows from the identity
\[
(a^2 - b^2)(c^2 - d^2) = (ac - bd)^2 - (bc - ad)^2.
\]

II. Properties of the Absolute Value

1. For \( a, b \in \mathbb{R} \) let \( \max(a, b) \) and \( \min(a, b) \) be the maximum and the minimum of \( a \) and \( b \), respectively. For any \( x \in \mathbb{R} \) let \( |x| = \sqrt{x^2} \) be the absolute value. Verify the following identities and inequalities:
\[
\max(x, -x) = |x|, \quad \min(x, -x) = -|x|, \quad x \leq |x| \quad \text{and} \quad -x \leq |x|.
\]
Convert the function \( x \mapsto |x| \) into a piecewise linear function.

Solution. The first part follows from the facts that \( x = |x| \) if \( x \geq 0 \) and \( -x = |x| \) if \( x \leq 0 \) so that \( \max(x, -x) = |x| \), \( \min(x, -x) = -|x| \), \( x \leq |x| \) and \( -x \leq |x| \). The inequalities follow from the trivial statements: \( a \leq \max(a, b) \) and \( b \leq \max(a, b) \). Finally,
\[
|x| = \begin{cases} 
  x, & \text{if } x \geq 0; \\
  -x, & \text{if } x < 0.
\end{cases}
\]

Remark: Alternatively, the domains \( x > 0 \) and \( x \leq 0 \) are also possible.

2. For any \( x \in \mathbb{R} \) let \( |x| = \sqrt{x^2} \) be the absolute value. Prove the following three fundamental properties of the absolute value:
   (a) Positivity: \( |x| \geq 0 \) for all \( x \in \mathbb{R} \) and \( |x| = 0 \) if and only if \( x = 0 \).
   (b) Homogeneity: For all \( \lambda, x \in \mathbb{R} \): \( |\lambda x| = |\lambda||x| \) (\( = |\lambda| \cdot |x| \)).
   (c) Convexity (triangle inequality): For all \( x, y \in \mathbb{R} \): \( |x + y| \leq |x| + |y| \).

Supplement: Prove that the function \( f(x) = |x| \) is indeed a convex function, that is for all \( x, y \in \mathbb{R} \) and all \( t \in [0, 1] \) one has
\[
|tx + (1 - t)y| \leq t|x| + (1 - t)|y|.
\]

Deduce from (2) the triangle inequality as a special case.
Solution.

(a) Positivity: By the definition of the square root $|x| \geq 0$ for all $x \in \mathbb{R}$. Obviously and $|0| = 0$ and for all $x \neq 0$ we have $|x| > 0$ since it is either $x \neq 0$ or $-x \neq 0$.

(b) Homogeneity: For all $\lambda, x \in \mathbb{R}$: $|\lambda x| = \sqrt{\lambda^2 x^2} = \sqrt{\lambda^2} \sqrt{x^2} = |\lambda| \cdot |x|$.

(c) Convexity (triangle inequality): For all $x, y \in \mathbb{R}$: $|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy$. Since $a \leq |a|$ for all real $a$ we have $xy \leq |xy|$ and by homogeneity $|xy| = |x| \cdot |y|$ from which

$$|x + y|^2 \leq x^2 + y^2 + 2|x| \cdot |y| = |x|^2 + |y|^2 + 2|x| \cdot |y| = (|x| + |y|)^2.$$

Taking the square root the triangle inequality follows. However, we prefer the following argument since it is more elementary. We have

$$0 \leq 2|x| \cdot |y| - 2xy = (|x| + |y|)^2 - |x + y|^2$$

$$= (|x| + |y| + |x + y|) \cdot (|x| + |y| - |x + y|)$$

Since the first factor is always strictly positive (except if $x = y = 0$ and in this case the triangle inequality is trivial) the second factor must be greater or equal than zero which proves the triangle inequality.

Supplement: To prove convexity we repeat the above argument. For all $x, y \in \mathbb{R}$ and all $t \in [0, 1]$ one has

$$0 \leq 2t(1-t)|x| \cdot |y| - 2t(1-t)xy$$

$$= (t|x| + (1-t)|y|)^2 - (t^2 x^2 + (1-t)^2 y^2 + 2t(1-t)xy)$$

$$= (t|x| + (1-t)|y|)^2 - (tx + (1-t)y)^2$$

$$= (t|x| + (1-t)|y| + |tx + (1-t)y|) \cdot ((t|x| + (1-t)|y| - |tx + (1-t)y|))$$

and we conclude as above. Choosing $t = \frac{1}{2}$ yields the triangle inequality.

3. Prove that for all nonzero reals $x, y$ the absolute value has the following symmetry property:

$$\frac{y}{|y|} - |y|x = \frac{x}{|x|} - |x|y.$$ (3)

Solution. Taking the square (note that $|a|^2 = a^2$ for all reals $a$) we have

$$\left| \frac{y}{|y|} - |y|x \right|^2 = 1 + x^2 y^2 - 2xy = \left| \frac{x}{|x|} - |x|y \right|^2.$$

In particular, since $1 + x^2 y^2 - 2xy = (1 - xy)^2$ it also follows that

$$\left| \frac{y}{|y|} - |y|x \right| = \left| \frac{x}{|x|} - |x|y \right| = |1 - xy|.$$

4. For all $x, y \in \mathbb{R}$ prove the following min-max identities for the absolute value:

$$|x + y| + |x - y| = |x| + |y| + |x| - |y| = 2 \max(|x|, |y|),$$

$$|x + y| - |x - y| = |x| + |y| - |x| - |y| = 2 \min(|x|, |y|).$$ (4)
Solution. An efficient method to solve many problems is to reduce the number of cases to analyse by a “symmetry argument”. In the present case, we note that the identities (4) remain unchanged under the replacements $x \mapsto -x$ and $y \mapsto -y$ since $|a| = |-a|$. Therefore it is sufficient to prove the identities (4) in the case $x, y \geq 0$. However, in this case $|x| = x$, $|y| = y$ etc. and the identities (4) are obviously true.
III. Applications of the Absolute Value

1. For \( x, y \in \mathbb{R} \) let \( \max(x, y) \) and \( \min(x, y) \) denote the maximum and the minimum of \( x \) and \( y \), respectively. Show that

\[
\max(x, y) = \frac{1}{2} (x + y + |x - y|),
\]

\[
\min(x, y) = \frac{1}{2} (x + y - |x - y|).
\]

Conclude that \( |\min(x, y)| \geq \min(|x|, |y|) \) and \( |\max(x, y)| \leq \max(|x|, |y|) \).

**Solution.** Without loss of generality we may assume \( y \geq x \). Then

\[
x = \min(x, y) = \frac{1}{2} (x + y - (y - x)) = \frac{1}{2} (x + y - |x - y|)
\]

and similarly for \( y = \max(x, y) \). Finally, since \( \min(|x|, |y|) = |x| \) if \( |x| \leq |y| \) and \( \min(|x|, |y|) = |y| \) if \( |x| \geq |y| \) it always follows that \( |x| = |\min(x, y)| \geq \min(|x|, |y|) \) and similarly for the inequality \( \max(x, y) \leq \max(|x|, |y|) \). Alternatively one could apply by the triangle inequality and the (5) to get

\[
|x| = |\min(x, y)| = \frac{1}{2} |x + y - |x - y|| \geq \frac{1}{2} |x + y| - |x - y| = \min(|x|, |y|).
\]

2. For \( x \in \mathbb{R} \) let \( T(x) = \max(1 - |x|, 0) \) be the "hat function". Convert \( T \) into a piecewise defined function. Show that

\[
T(x) = \frac{1}{2} (|x - 1| + |x + 1| - 2|x|).
\]

**Solution.**

\[
T(x) = \begin{cases} 
0, & \text{if } x < -1; \\
1 + x, & \text{if } -1 \leq x < 0; \\
1 - x, & \text{if } 0 \leq x < 1; \\
0, & \text{if } 1 \leq x.
\end{cases}
\]

and since \( \max(a, b) = \frac{1}{2} (a + b + |a - b|) \) for any \( a, b \in \mathbb{R} \) it follows from the identities (5) that

\[
T(x) = \frac{1}{2} (1 - |x| + |1 - |x||) = \frac{1}{2} (1 - |x| + |1 - x| + |1 + x| - 1 - |x|)
\]

proving the assertion.