I. Squares, Square Roots

1. Suppose \( a \geq b \geq 0 \) and \( c \geq d \geq 0 \). Prove that

\[
\sqrt{a^2 - b^2} \sqrt{c^2 - d^2} \leq ac - bd.
\]

II. Properties of the Absolute Value

1. For \( a, b \in \mathbb{R} \) let \( \max(a, b) \) and \( \min(a, b) \) be the maximum and the minimum of \( a \) and \( b \), respectively. For any \( x \in \mathbb{R} \) let \( |x| = \sqrt{x^2} \) be the absolute value. Verify the following identities and inequalities: \( \max(x, -x) = |x| \), \( \min(x, -x) = -|x| \), \( x \leq |x| \) and \( -x \leq |x| \). Convert the function \( x \mapsto |x| \) into a piecewise linear function.

2. For any \( x \in \mathbb{R} \) let \( |x| = \sqrt{x^2} \) be the absolute value. Prove the following three fundamental properties of the absolute value:

(a) Positivity: \( |x| \geq 0 \) for all \( x \in \mathbb{R} \) and \( |x| = 0 \) if and only if \( x = 0 \).

(b) Homogeneity: For all \( \lambda, x \in \mathbb{R} \): \( |\lambda x| = |\lambda||x| \) (\( = |\lambda| \cdot |x| \)).

(c) Convexity (triangle inequality): For all \( x, y \in \mathbb{R} \): \( |x + y| \leq |x| + |y| \).

Supplement: Prove that the function \( f(x) = |x| \) is indeed a convex function, that is for all \( x, y \in \mathbb{R} \) and all \( t \in [0, 1] \) one has

\[
|tx + (1 - t)y| \leq t|x| + (1 - t)|y|.
\] (2)

Deduce from (2) the triangle inequality as a special case.

3. Prove that for all nonzero reals \( x, y \) the absolute value has the following symmetry property:

\[
\frac{y}{|y|} - |y|x = \frac{x}{|x|} - |x|y. \tag{3}
\]

4. For all \( x, y \in \mathbb{R} \) prove the following min-max identities for the absolute value:

\[
|x + y| + |x - y| = |x| + |y| + |x| - |y| = 2 \max(|x|, |y|),
\]

\[
|x + y| - |x - y| = |x| + |y| - |x| - |y| = 2 \min(|x|, |y|). \tag{4}
\]
III. Applications of the Absolute Value

1. For $x, y \in \mathbb{R}$ let $\max(x, y)$ and $\min(x, y)$ denote the maximum and the minimum of $x$ and $y$, respectively. Show that

$$\max(x, y) = \frac{1}{2}(x + y + |x - y|),$$

$$\min(x, y) = \frac{1}{2}(x + y - |x - y|).$$

Conclude that $|\min(x, y)| \geq \min(|x|, |y|)$ and $|\max(x, y)| \leq \max(|x|, |y|)$.

2. For $x \in \mathbb{R}$ let $T(x) = \max(1 - |x|, 0)$ be the "hat function". Convert $T$ into a piecewise defined function. Show that

$$T(x) = \frac{1}{2}(|x - 1| + |x + 1| - 2|x|).$$