▶ Setting

The goal of this project is to develop a CP-based method for solving a linear system of the form

$$(I \otimes I \otimes A + I \otimes A \otimes I + A \otimes I \otimes I)x = b,$$

where A is a tridiagonal symmetric positive definite matrix of the form

$$A = (n+1)^2 \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Letting $\mathcal{X}, \mathcal{B} \in \mathbb{R}^{n \times n \times n}$ be such that $x = \text{vec}(\mathcal{X})$ and $b = \text{vec}(\mathcal{B})$, this linear system is equivalent to

$$A \circ_1 \mathcal{X} + A \circ_2 \mathcal{X} + A \circ_3 \mathcal{X} = \mathcal{B}. \tag{1}$$

We will look at two different right-hand sides:

1. $\mathcal{B}_1(i_1, i_2, i_3) = \sin(\xi(i_1) + \xi(i_2) + \xi(i_3))$,
2. $\mathcal{B}_1(i_1, i_2, i_3) = \sqrt{\xi(i_1)^2 + \xi(i_2)^2 + \xi(i_3)^2}$,

with $\xi(i) = i/(n+1)$.

▶ Tasks

1. Implement the ALS method for approximating a tensor in CP decomposition. Apply it to $\mathcal{B}_1$ and $\mathcal{B}_2$ for $n = 200$ with the tensor rank chosen such that the norm of the error stays below $10^{-4}\|\mathcal{B}_i\|$.
2. Implement a reference method for solving (1) for small n, e.g., using `kron` and backslash. What is the largest value of n you can handle this way?
3. Develop and implement a low-rank solver for (1) by applying ALS to the optimization problem

$$\min_{\mathcal{X} \text{ in CP decomposition}} \frac{1}{2}\langle \mathcal{X}, A \circ_1 \mathcal{X} + A \circ_2 \mathcal{X} + A \circ_3 \mathcal{X} \rangle - \langle \mathcal{X}, \mathcal{B} \rangle,$$

   where $\mathcal{B}$ is in CP decomposition. Develop a heuristics for adapting the tensor rank of $\mathcal{X}$. Apply your method with $n = 200$ for $\mathcal{B}_1$ and $\mathcal{B}_2$. What is the largest value of n you can handle this way?