▶ Setting

The goal of this project is to understand adaptive methods for randomized SVDs and combine them with the HOSVD.

▶ Tasks

1. Prove Lemma 4.1 of [N Halko, PG Martinsson, JA Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. SIAM review, 2011], which is the lemma on Slide 17 from Lecture 2. Suggestion for how to proceed with the proof:
   - Use the following result[1]: For $Z \sim \chi_1^2$ it holds that $\mathbb{P}(Z < \mu) \leq \sqrt{2/\pi}\sqrt{\mu}$ for any $\mu > 0$.
   - Prove:[2] For a Gaussian random vector x and a symmetric positive semi-definite matrix B with largest eigenvalue λ_1, it holds that
   $$\mathbb{P}(\theta^2 x^T B x < \lambda_1) \leq \sqrt{2/\pi}\theta^{-1}.$$
   - For a general matrix A, prove that
   $$\mathbb{P}(\sqrt{2/\pi}\alpha\|Ax\|_2 < \|A\|_2) \leq \alpha^{-1}.$$
   - Conclude the result.[3]
2. Implement a variant the randomized SVD from Slide 2 of Lecture 2 that chooses the rank adaptively such that $\|(I - QQ^T)A\|_2$ is smaller than a prescribed user tolerance ε.
3. Combine the randomized SVD with HOSVD for compressing a third-order tensor in Tucker decomposition. Apply it to the $n \times n \times n$ tensor given by $\mathcal{B}_1(i_1, i_2, i_3) = \sin(\xi(i_1) + \xi(i_2) + \xi(i_3))$, with $\xi(i) = i/(n+1)$, for $n = 200$.
4. Develop and prove a variant of the theorem from Slide 26 of Lecture 5, which takes the additional error from the randomized SVD into account.

[1] You do not need to prove this result

[2] Following, e.g. [Dixon, Estimating extremal eigenvalues and condition numbers of matrices, SIAM J. Numer. Anal. 20 (1983) 812–814

[3] Using techniques found, e.g., in Section 3.4 of [F. Woolfe, E. Liberty, V. Rokhlini, and M. Tygert, A fast randomized algorithm for the approximation of matrices, Appl. Comput. Harmon. Anal., 25 (2008), pp. 335–366].