Unspanned Stochastic Volatility in the Multi-factor CIR Model*

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Abstract

We provide necessary and sufficient conditions for a multi-factor Cox-Ingersoll-Ross (CIR) model to exhibit unspanned stochastic volatility (USV). We then construct a class of three-factor CIR models that exhibit USV. This clarifies the to date open question whether multi-factor CIR models can exhibit USV or not.

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1 Introduction

Empirical evidence suggests that fixed income markets exhibit unspanned stochastic volatility (USV), i.e. that one cannot fully hedge volatility risk

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solely using a portfolio of bonds, see [1, 5, 6, 4]. One of the basic models for the term structure of interest rates is the multi-factor Cox-Ingersoll-Ross (CIR) model. It has been unknown to date whether multi-factor CIR models can exhibit USV or not. In our paper we first give necessary and sufficient conditions for a multi-factor CIR model to exhibit USV. These conditions are knife-edge and reveal that multi-factor CIR models do not exhibit USV in general. In particular, we show that the number of USV factors in a $d$-factor CIR model is limited by $d - 2$, so that there exists no two-factor CIR model that exhibits USV. We then construct a class of three-factor CIR models that exhibit USV. This clarifies the to date open question whether multi-factor CIR models can exhibit USV or not.

The structure of the paper is as follows. In Section 2 we introduce the multi-factor CIR models and formally define USV. In Section 3 we give equivalent conditions for USV. In Section 4 we construct a class of three-factor CIR models that exhibit USV.

2 Multi-factor CIR Models and USV

Throughout we fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ where $\mathbb{Q}$ denotes the risk-neutral pricing measure. We briefly recap the definition of the multi-factor CIR model, see e.g. [3] for details. Let $d \in \mathbb{N}$. The $d$-factor CIR model consists of the $\mathbb{R}^d_+$-valued square-root diffusion factor process $X_t$ with dynamics of the form

\[ dX_t = (b + \beta X_t)dt + \text{diag}(\sigma_1 \sqrt{X_{1t}}, \ldots, \sigma_d \sqrt{X_{dt}})dW_t, \tag{1} \]

for some $b \in \mathbb{R}^d_+$, $\beta \in \mathbb{R}^{d \times d}$ with nonnegative off-diagonals, $\beta_{ij} \geq 0$ for $i \neq j$, $\sigma_i \geq 0$, and where $W_t$ is a $d$-dimensional Brownian motion. The short rate is given by

\[ r_t = \rho^\top X_t \tag{2} \]

for some parameter $\rho \in \mathbb{R}^d_+ \setminus \{0\}$. The price at time $t$ of a zero-coupon bond maturing at time $T$, given by

\[ P(t, T) = \mathbb{E}[e^{-\int_t^T r_s ds} | \mathcal{F}_t], \]

becomes an exponential affine function of the factors $X_t$,

\[ P(t, T) = e^{-A(T-t) - B(T-t)^\top X_t}. \]
The \( \mathbb{R} \)- and \( \mathbb{R}^d \)-valued functions \( A(\tau) \) and \( B(\tau) \) solve the Riccati equations

\[
A'(\tau) = b^\top B(\tau), \quad A(0) = 0, \\
B'_i(\tau) = -\frac{\sigma_i^2}{2} B_i(\tau)^2 + \beta_i^\top B(\tau) + \rho_i, \quad B(0) = 0,
\]

for \( i = 1, \ldots, d \) and where \( \beta_i \) denotes the \( i \)-th column of \( \beta \). Note that \( \rho = B'(0) \).

We now define the concept of unspanned stochastic volatility (USV). The \( d \)-factor CIR model (1)–(2) can exhibit USV if there are directions \( \xi \) in \( \mathbb{R}^d \) such that the term structure of bond prices \( P(t, T), T \geq t \), does not change when \( X_t \) moves along \( \xi \). Such directions are thus unspanned by the term structure but could still feed into the volatility of \( X_t \). Unspanned directions can be revealed by a change of coordinates. This is formalized as follows. Let \( S \) be a bijective linear transformation on \( \mathbb{R}^d \) and write \( \tilde{X}_t = SX_t = (Z_t; U_t) \) with \( Z_t \in \mathbb{R}^m \) and \( U_t \in \mathbb{R}^n \) for some \( m, n \in \mathbb{N} \) such that \( m + n = d \). Then the zero-coupon bond prices can be rewritten as

\[
P(t, T) = e^{-A(T-t) - \tilde{B}_Z(T-t)^\top Z_t - \tilde{B}_U(T-t)^\top U_t}
\]

for \( \tilde{B}(\tau) = S^{-\top} B(\tau) = (\tilde{B}_Z(\tau); \tilde{B}_U(\tau)) \) corresponding to \( \tilde{X}_t = (Z_t; U_t) \).

**Definition 2.1.** The \( d \)-factor CIR model (1)–(2) exhibits USV if there exists a bijective linear transformation \( S \) on \( \mathbb{R}^d \) such that

(i) \( \tilde{B}_U(\tau) = 0 \) for all \( \tau \geq 0 \);

(ii) \( Z_t \) is not autonomous;

Condition (i) implies that the term structure at time \( t \) is a function of the term structure factors \( Z_t \) only and does not directly depend on \( U_t \),

\[
P(t, T) = e^{-A(T-t) - \tilde{B}_Z(T-t)^\top Z_t}.
\]

Condition (ii) states that the \( \mathcal{F}_t \)-conditional distribution of \( Z_T \), for \( T > t \), and thus of future bond prices, depends on \( U_t \). That is, the USV factors \( U_t \) feeds into the martingale characteristics (volatility) of \( Z_t \).
3 Equivalent Conditions for Exhibiting USV

We derive equivalent conditions to (i) and (ii) that can be easily verified for a given CIR model. We first define the map

\[ H : \mathbb{R}^d \to \mathbb{R}^d, \quad H(v) = -\frac{1}{2}\sigma^2 \circ v \circ v + \beta^\top v + \rho, \]

where \( \circ \) denotes component-wise multiplication (Hadamard product) and \( \sigma^2 := (\sigma_1^2, \ldots, \sigma_d^2)^\top \). For any \( v \in \mathbb{R}^d \), let \( \phi(\tau, v) \) and \( \psi(\tau, v) \) be the solution of the following system of Riccati differential equations:

\[
\begin{align*}
\phi'(\tau, v) &= b^\top \psi(\tau, v), \quad \phi(0, v) = 0, \\
\psi'(\tau, v) &= H(\psi(\tau, v)), \quad \psi(0, v) = v.
\end{align*}
\]

It follows by inspection that \( \psi(\tau, 0) = B(\tau) = S^\top \tilde{B}(\tau) \) and hence Condition (i) is equivalent to

(i) \( \psi(\tau, 0) = B(\tau) \in S^\top (\mathbb{R}^m \times \{0\}^n) \) for all \( \tau \geq 0 \).

Condition (i) also implies that \( \rho = B'(0) = \lim_{\tau \to 0} B(\tau)/\tau \in S^\top (\mathbb{R}^m \times \{0\}^n) \).

To rewrite Condition (ii) we consider following lemma:

**Lemma 3.1.** The process \( Z_t \) is autonomous if and only if

\[
H(S^\top (\mathbb{R}^m \times \{0\}^n)) \subseteq S^\top (\mathbb{R}^m \times \{0\}^n).
\]

*Proof.* Let \( z \in \mathbb{R}^m, u \in \mathbb{R}^n \) and \( x \in \mathbb{R}^d \) such that \( Sx = (z; u) \). Then, for any \( v \in \mathbb{R}^m \) and \( t \geq 0 \) such that the left-hand side is finite,

\[
E_{(z, u)} \left[e^{v^\top Z_t}\right] = E_x \left[e^{(S^\top v)^\top X_t}\right] = e^{\phi(t, S^\top (v)) + \psi(t, S^\top (v))^\top x},
\]

where in the last equality we apply the affine property of \( X_t \), see e.g. [2].

For any \( v \in \mathbb{R}^m \), there is an open interval \( I \subset \mathbb{R} \) containing zero such that (5) holds for all \( t \in I \). If \( Z_t \) is autonomous, then the last quantity in (5) does not depend on \( u \), hence

\[
\psi(t, S^\top \begin{pmatrix} v \\ 0 \end{pmatrix}) \in S^\top (\mathbb{R}^m \times \{0\}^n), \quad t \in I,
\]

which implies \( \psi'(0, S^\top \begin{pmatrix} v \\ 0 \end{pmatrix}) = H(S^\top \begin{pmatrix} v \\ 0 \end{pmatrix}) \in S^\top (\mathbb{R}^m \times \{0\}^n) \), as required.
Conversely, if (4) holds, then $\psi(t, S^T(v_0))$ lies in $S^T(\mathbb{R}^m \times \{0\}^n)$ for all $t \geq 0$ and $v \in \mathbb{R}^m$ such that this quantity exists. Consequently the left-hand side of (5) does not depend on $u$, and the distribution of $Z_t$ is a function of $z$ only. This shows that $Z_t$ is an autonomous process. \qed

Lemma 3.1 implies that Condition (ii) is equivalent to

(ii) $\exists v \in \mathbb{R}^m$ such that

$$H(S^T\begin{pmatrix} v \\ 0 \end{pmatrix}) \notin S^T(\mathbb{R}^m \times \{0\}^n).$$

Lemma 3.1 also yields an important result which shows that a CIR model needs at least two term structure factors in order to exhibit USV.

**Theorem 3.2.** Whether the $d$-factor CIR model (1)–(2) exhibits USV or not depends on the model parameters $\sigma^2$, $\beta$, and $\rho$ only. There can be at most $d - 2$ USV factors, so that necessarily the number of term structure factors satisfies $m \geq 2$.

**Proof.** The first statement follows from the above equivalent conditions (i) and (ii). For the second statement, we argue by contradiction and suppose that the $d$-factor CIR model exhibits USV with $m = 1$. Condition (i) implies that

$$S^T(\mathbb{R} \times \{0\}^{d-1}) = \text{span}(\rho)$$

and $\{B(\tau) \mid \tau \geq 0\} \supset \{s\rho \mid s \in I\}$ for some open interval $I \subset \mathbb{R}$ containing zero. Let $\xi \perp S^T(\mathbb{R} \times \{0\}^{d-1})$. Then

$$\xi^T H(B(\tau)) = \xi^T B'(\tau) = 0 \quad \text{for all } \tau \geq 0$$

and hence $\xi^T H(s\rho) = 0$ for all $s \in I$. Since $H(w)$ is an analytic function in $w \in \mathbb{R}^d$ we conclude that $\xi^T H(s\rho) = 0$ for all $s \in \mathbb{R}$ and hence (4), which contradicts Condition (ii). \qed

A simple consequence is stated in the following corollary.

**Corollary 3.3.** There exists no two-factor CIR model that exhibits USV.

An example of an alternative two-factor Markov model of the term structure that exhibits USV is given in [4, Section II].
4 A Class of Three-Factor CIR Models that Exhibit USV

We construct a class of three-factor CIR models that exhibit USV. Corollary 3.3 indicates that the dimension $d = 3$ is the first non trivial case that can be considered. Following Theorem 3.2, we aim at constructing a model with $m = 2$ term structure factors and $n = 1$ unspanned factor. Let us then choose the following bijective transformation matrix $S$:

$$S := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

The factor process is of the form

$$SX_t = \begin{pmatrix} Z_t \\ U_t \end{pmatrix} = \begin{pmatrix} X_1 + X_3 \\ X_2 + X_3 \\ X_3 \end{pmatrix}. \quad (7)$$

Note that the projection of $SR^3$ onto $\mathbb{R}^2 \times \{0\}$ is $\mathbb{R}_+^2 \times \{0\}$, so that the term structure factor process $Z_t$ is $\mathbb{R}_+^2$-valued.

Here is our main result.

**Theorem 4.1.** Any three-factor CIR model with $\sigma_i = \sqrt{2}$, $i = 1, 2, 3$,

$$\beta = \begin{pmatrix} \beta_{11} & 0 & \beta_{13} \\ 0 & \beta_{22} & \beta_{23} \\ 0 & 0 & \beta_{33} \end{pmatrix},$$

for parameters

$$\beta_{22} < \beta_{11} < 0, \quad \beta_{23} > 0,$$

$$\beta_{13} = \frac{8\rho_2}{\beta_{11} - \beta_{22}} + \beta_{23} - 2\beta_{22}, \quad (7)$$

$$\beta_{33} = \beta_{11} + \beta_{22} - \frac{1}{2}(\beta_{13} + \beta_{23}),$$

and

$$\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_1 + \rho_2 \end{pmatrix}, \quad (8)$$
for parameters $\rho_2 > 0$ and

$$\rho_1 = \frac{1}{8}(\beta_{11} - \beta_{22})(\beta_{13} - \beta_{23} - 2\beta_{11}),$$

(9)

exhibits USV.

Note that after component-wise scaling of the factors we can always normalise $\sigma_i = \sqrt{2}$, $i = 1, 2, 3$, without loss of generality. Moreover, (7) and (8) imply that $\beta_{13} > 0$, $\beta_{33} < 0$, and $\rho_1 > 0$. Therefore, the corresponding CIR model is well defined and mean-reverting since the diagonal elements (eigenvalues) of $\beta$ are negative. While Theorem 4.1 gives a parametric class of three-factor CIR models that exhibit USV, with four free parameters ($\beta_{11}, \beta_{22}, \beta_{23}, \rho_2$), the parameter constraints (7)–(9) are knife-edge and difficult to find. This is in contrast to the linear-rational term structure models introduced in [4] that generically can exhibit USV. \(^1\)

**Proof.** We must verify that Conditions (i) and (ii) are satisfied, with $B$ being the solution to (3). The obvious relation

$$S^\top(\mathbb{R}^2 \times \{0\}) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{span}\{w\}^\perp,$$

(10)

with $w = (1, 1, -1)^\top$, shows that Condition (i) holds if and only if $B$ is of the form

$$B(\tau) = \begin{pmatrix} B_1(\tau) \\ B_2(\tau) \\ B_1(\tau) + B_2(\tau) \end{pmatrix}, \quad \tau \geq 0.$$  

Condition (i) thus follows once we prove that

$$B_3(\tau) = B_1(\tau) + B_2(\tau), \quad \tau \geq 0.$$  

(11)

The assumed structure of $\beta$ and $\rho$ allows us to rewrite (3) as

$$B_1'(\tau) = -B_1(\tau)^2 + \beta_{11}B_1(\tau) + \rho_1,$$

(12)

$$B_2'(\tau) = -B_2(\tau)^2 + \beta_{22}B_2(\tau) + \rho_2,$$

(13)

$$B_3'(\tau) = -B_3(\tau)^2 + \beta_{13}B_1(\tau) + \beta_{23}B_2(\tau) + \beta_{33}B_3(\tau) + \rho_1 + \rho_2,$$

(14)

\(^1\)The drift constraints in the linear-rational square-root (LRSQ) model in [4, Section II] are straightforward such that the transformed process $Z_t$ has an autonomous drift.
along with the initial condition \( B_1(0) = B_2(0) = B_3(0) = 0 \). Uniqueness of solutions to this system yields (11) once we prove that \( B_1 + B_2 \) satisfies (14), that is,

\[
B'_1(\tau) + B'_2(\tau) = -(B_1(\tau) + B_2(\tau))^2 + \beta_{13}B_1(\tau) \\
+ \beta_{23}B_2(\tau) + \beta_{33}(B_1(\tau) + B_2(\tau)) + \rho_1 + \rho_2, \quad \tau \geq 0.
\]

In view of (12) and (13), this is equivalent to

\[
c_1 B_1(\tau) + c_2 B_2(\tau) - 2B_1(\tau)B_2(\tau) = 0, \quad \tau \geq 0,
\]

where

\[
c_1 := \beta_{13} + \beta_{33} - \beta_{11}, \quad c_2 := \beta_{23} + \beta_{33} - \beta_{22}.
\]

To prove (15), we use that the solutions to (12) and (13) are explicitly known,

\[
B_1(\tau) = \frac{2\rho_1(e^{\theta_1\tau} - 1)}{\theta_1(e^{\theta_1\tau} + 1) - \beta_{11}(e^{\theta_1\tau} - 1)}, \quad \theta_1 := \sqrt{\beta_{11}^2 + 4\rho_1},
\]

\[
B_2(\tau) = \frac{2\rho_2(e^{\theta_2\tau} - 1)}{\theta_2(e^{\theta_2\tau} + 1) - \beta_{22}(e^{\theta_2\tau} - 1)}, \quad \theta_2 := \sqrt{\beta_{22}^2 + 4\rho_2},
\]

see e.g. [3]. The form (7) and (9) of \( \beta \) and \( \rho \) implies that \( \theta := \theta_1 = \theta_2 \), and in order to simplify notation we write \( B_i(\tau) = N_i(\tau)/D_i(\tau) \) with

\[
N_i(\tau) := 2\rho_i(e^{\theta_i\tau} - 1), \\
D_i(\tau) := \theta(e^{\theta_i\tau} + 1) - \beta_{ii}(e^{\theta_i\tau} - 1).
\]

With this notation, (15) can equivalently be written

\[
c_1 N_1(\tau)D_2(\tau) + c_2 N_2(\tau)D_1(\tau) - 2N_1(\tau)N_2(\tau) = 0, \quad \tau \geq 0,
\]

which upon inserting the expressions for \( N_i(\tau) \) and \( D_i(\tau) \) becomes

\[
-\gamma_0 + \gamma_1 e^{\theta\tau} + (\gamma_0 - \gamma_1)e^{2\theta\tau} = 0, \quad \tau \geq 0,
\]

where

\[
\gamma_0 := 2c_1\rho_1(\theta + \beta_{22}) + 2c_2\rho_2(\theta + \beta_{11}) + 8\rho_1\rho_2, \\
\gamma_1 := 4c_1\rho_1\beta_{22} + 4c_2\rho_2\beta_{11} + 16\rho_1\rho_2.
\]
A further calculation shows that $\gamma_0 = \gamma_1 = 0$ holds if
\begin{align*}
\beta_{13} + \beta_{23} &= 2(\beta_{11} + \beta_{22} - \beta_{33}), \\
(\beta_{13} - \beta_{23})(\beta_{11} - \beta_{22}) &= (\beta_{11} - \beta_{22})^2 + 4(\rho_1 + \rho_2).
\end{align*}
This system is indeed satisfied by the model parameters $\beta$ and $\rho$ in (7) and (9). We conclude that (16), hence (15), and therefore also Condition (i), is satisfied.

It remains to check Condition (ii), namely that
\[
w^\top H(S^\top \begin{pmatrix} v \\ 0 \end{pmatrix}) \neq 0
\]
for some $v = (v_1, v_2)^\top \in \mathbb{R}^2$, see (10). In our case we have
\[
w^\top H(S^\top \begin{pmatrix} v \\ 0 \end{pmatrix}) = 2v_1v_2 + v_1(\beta_{11} - \beta_{13} - \beta_{33}) + v_2(\beta_{22} - \beta_{23} - \beta_{33}),
\]
which is certainly nonzero for some $v \in \mathbb{R}^2$.

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