ON THE EIGENVALUE DECAY OF SOLUTIONS TO OPERATOR LYAPUNOV EQUATIONS

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Abstract. This paper is concerned with the eigenvalue decay of the solution to operator Lyapunov equations with right-hand sides of finite rank. We show that the kth (generalized) eigenvalue decays exponentially in $\sqrt{k}$, provided that the involved operator A generates an exponentially stable analytic semigroup, and A is either self-adjoint or diagonalizable with its eigenvalues contained in a strip around the real axis. Numerical experiments with discretizations of 1D and 2D PDE control problems confirm this decay.

1. Introduction

The Lyapunov matrix equation

$$AX + XA^T = -BB^T \tag{1.1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ plays a central role in balanced truncation model reduction for linear time-invariant control systems [1]. Assuming that A is stable (i.e., all its eigenvalues have negative real part), the equation (1.1) has a unique, bounded, nonnegative, and self-adjoint solution X. Typically, the eigenvalues of X decay very quickly when the right-hand side has low rank, that is, $m \ll n$. This decay property is strongly linked to the approximation error attained by balanced truncation as well as the performance of low-rank methods for solving (1.1). Consequently, a number of works [2, 8, 9, 13, 16, 18] have studied this decay and derived a priori estimates.

By now, the situation is fairly well understood for a symmetric negative definite matrix A. In this case, it can be shown [18, 13] that there is a matrix X_k of rank km such that

$$\|X - X_k\|_F \leq \frac{8\|B\|_F}{|\lambda_{\max}(A)|} \exp\left(\frac{-k\pi^2}{\log(8\,\kappa(A))}\right), \tag{1.2}$$

where $\lambda_{\max}(A)$ denotes the largest eigenvalue and $\kappa(A)$ the condition number of A. By the Eckart-Young theorem, this estimate implies that the sorted eigenvalues $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ of X decay exponentially:

$$\lambda_k(X) \lesssim \gamma^k \quad \text{with} \quad \gamma = \exp\left(\frac{-\pi^2}{m\log(8\,\kappa(A))}\right). \tag{1.3}$$

This bound bears the disadvantage that it deteriorates as $\kappa(A) \to \infty$, a situation of practical relevance when A comes from the (increasingly refined) discretization of an unbounded operator. Indeed, the numerical calculations for an example in Section 5 seem to indicate that the *exponential* decay property gets lost as $\kappa(A) \to \infty$. In fact, the decay is observed to be exponential with respect to $\sqrt{k}$, instead of k. We analyze the generalized eigenvalues of X in the scale of Hilbert spaces associated to A. The aim of this paper is to prove this property for the underlying operator Lyapunov equation, when A has eigenvalues contained in a strip around the real axis

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and is diagonalizable, and B has finite rank. Our result extends related work by Opmeer [14], which implies superpolynomial decay.

2. Preliminaries

In this section, we will formalize the notation and point out some of the conventions that will be used in this paper.

Given a Gelfand triple $\mathcal{X} \subset \mathcal{H} \subset \mathcal{Z}$ of Hilbert spaces, where $\mathcal{Z} = \mathcal{X}'$ is the dual space to $\mathcal{X}$, we consider a bounded operator A from $\mathcal{X}$ to $\mathcal{Z}$ with a bounded inverse. We let $A' : \mathcal{Z}' \to \mathcal{X}'$ denote the dual operator to A in the duality paring $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathcal{Z} \times \mathcal{X}}$. After identifying biduals we may also write $A' : \mathcal{X} \to \mathcal{Z}$. We will use the notation $\mathcal{X} = \mathrm{Dom}(A)$, since A can also be interpreted as an unbounded operator on $\mathcal{H}$, and then we will denote its domain of definition as $\mathrm{Dom}_{\mathcal{H}}(A) = \{u \in \mathcal{H} \ : \ \|Au\|_{\mathcal{H}} < \infty\}$. Moreover, we consider a (not necessarily bounded) linear operator $B : \mathcal{U} \to \mathcal{Z}$ for a Hilbert space $\mathcal{U}$ with inner product $\big(\cdot, \cdot\big)_{\mathcal{U}}$.

The operators A, B give rise to the *Lyapunov operator equation* in a linear operator X:

$$AX + XA' = -BB', \tag{2.1}$$

which formally stands for the variational formulation

$$\big\langle Xz_1, A'z_2 \big\rangle_{\mathcal{Z} \times \mathcal{X}} + \big\langle A'z_1, Xz_2 \big\rangle_{\mathcal{Z} \times \mathcal{X}} = \mathfrak{b}(z_1, z_2), \qquad z_1, z_2 \in \mathcal{X}. \tag{2.2}$$

with the sesquilinear form $\mathfrak{b}(z_1, z_2) := -\big(B'z_1, B'z_2\big)_{\mathcal{U}}$. We refer to, e.g., [6, 11, 19, 25] for a more detailed discussion of this equation.

Example 2.1 ([5])**.** Consider the point-wise control of a diffusion process on the interval $[0, 1]$:

$$z_t(t, x) = \kappa z_{xx}(t, x) + \delta(x - x_b)u(t), \qquad z(x, 0) \equiv 0, \tag{2.3}$$

$$y(t) = z(t, x_c), \qquad z(0, t) = z(1, t) = 0, \tag{2.4}$$

where $\kappa > 0$ is the diffusion coefficient and $0 < x_b < x_c < 1$. To set up the operator Lyapunov equation (2.1) for the controllability Gramian, we choose the usual Sobolev spaces $\mathcal{X} = H_0^1(0, 1)$, $\mathcal{H} = L^2(0, 1)$, and $\mathcal{Z} = H^{-1}(0, 1)$. Then $A = \partial_{xx}$ and B is defined by $B : u \mapsto u\ \delta(x - x_b)$ for $u \in \mathbb{R}$. $\diamond$

Let us assume that A is the infinitesimal generator of an exponentially stable analytic semigroup $(\exp(tA))_{t \geq 0}$ on $\mathcal{H}$. The results of [25, Chapter 5] imply the existence and uniqueness of a bounded nonnegative self-adjoint solution $X : \mathcal{H} \to \mathcal{H}$ to the Lyapunov equation (2.1), provided that A^{-1} is compact and $A^{-1}B$ is bounded. Furthermore, under the additional assumption that $A^{-1}B$ has finite rank Opmeer [14] has proved that X is not only bounded but also contained in every Schatten class [20].

2.1. **Choice of Hilbert spaces.** Instead of general Hilbert spaces $\mathcal{X}$ and $\mathcal{Z}$, we will use interpolation spaces associated with A. For this purpose, we work with the restricted operator $A : \mathrm{Dom}_{\mathcal{H}}(A) \subset \mathcal{X} \to \mathcal{H}$, which admits the adjoint A^*. Additionally we will assume that A possesses a Riesz basis of eigenvectors $\{\psi_i\}_{i \in \mathbb{N}}$ in $\mathcal{H}$ with associated eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$, this implies $\sup_{i \in \mathbb{N}} \operatorname{Re} \lambda_i < 0$. The Riesz property allows us to represent every $f \in \mathcal{H}$ as

$$f = \sum_{i \in \mathbb{N}} (f, \phi_i)\, \psi_i = \sum_{i \in \mathbb{N}} (f, \psi_i)\, \phi_i,$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathcal{H}$ and $\{\phi_i\}_{i \in \mathbb{N}}$ is a sequence of eigenvectors for A^*, normalized such that $(\phi_i, \psi_i) = 1$ for $i \in \mathbb{N}$.

Following [23] we define for every $\alpha \in \mathbb{R}$, the Hilbert space

$$\mathcal{H}_\alpha = \left\{ \sum_{i\in\mathbb{N}} f_i \psi_i : \; \{f_i |\lambda_i|^\alpha\}_{i\in\mathbb{N}} \in \ell^2 \right\}$$

with the scalar product

$$(f,g)_\alpha = \sum_{i\in\mathbb{N}} (f,\psi_i)(\psi_i,g)|\lambda_i|^{2\alpha}.$$

It holds $\mathcal{H}_{\alpha_1} \subset \mathcal{H} \subset \mathcal{H}_{\alpha_2}$ whenever $\alpha_2 \le 0 \le \alpha_1$. Analogously, we define the Hilbert space

$$\mathcal{H}^d_\alpha = \Big\{ \sum_{i\in\mathbb{N}} f_i \phi_i \; : \; \{f_i |\lambda_i|^\alpha\}_{i\in\mathbb{N}} \in l^2(\mathbb{N}) \Big\}$$

associated with the adjoint operator A^*.

Using this notation, $\mathcal{H} = \mathcal{H}_0 = \mathcal{H}^d_0$, $\mathrm{Dom}_{\mathcal{H}}(A) = \mathcal{H}_1$, $\mathrm{Dom}(A^*) = \mathcal{H}^d_1$, and we may set $\mathcal{Z} = \mathcal{H}_{-1/2}$. For Example 2.1, where A is the Dirichlet Laplace operator, we simply have $\mathcal{X} = H^1_0(0,1) = \mathcal{H}_{1/2}$ and $\mathcal{Z} = H^{-1}(0,1) = \mathcal{H}_{-1/2}$. The following example covers a more complicated situation.

Example 2.2. Let $A = \partial_x(a\partial_x) + b\partial_x + c$ for (possibly complex valued) functions $a, b, c \in L^\infty(0,1)$. Then Kato's square root theorem [3] yields $H^1_0(0,1) = \mathrm{Dom}_{\mathcal{H}}((-A)^{1/2})$. In the case that a is a real valued function such that there exists a Lipschitz function β with $\partial_x \beta = \frac{b}{2a}$ then A has real eigenvalues and is diagonalizable by the multiplication operator $Q : \psi \mapsto e^\beta \psi$, see [7], and so $H^1_0(0,1) = \mathcal{H}_{1/2} = \mathcal{H}^d_{1/2}$. $\diamond$

3. Selfadjoint case

We first consider the situation when A is self-adjoint on $\mathcal{H}$, has a compact resolvent and negative eigenvalues. We choose $\mathcal{Z} = \mathcal{H}_{-1/2}$, which is equipped with the scalar product $(\cdot, |A|^{-1}\cdot) = (|A|^{-1/2}\cdot, |A|^{-1/2}\cdot)$, and $\mathcal{X} = \mathcal{H}_{1/2} = \mathrm{Dom}_{\mathcal{H}}(|A|^{1/2})$.

Additionally, we assume that the product $|A|^{-1/2}B$ is bounded. This is equivalent to the assumption that

$$b(\psi,\phi) := -\mathfrak{b}(|A|^{-1/2}\psi, |A|^{-1/2}\phi)$$

is everywhere defined and bounded on $\mathcal{H}$. As discussed in [10], the substitutions $\psi = |A|^{1/2} z_1$ and $\phi = |A|^{1/2} z_2$ then allow us to turn (2.2) into the equivalent equation

$$\big(|A|^{1/2}\psi, X|A|^{-1/2}\phi) + (X|A|^{-1/2}\psi, |A|^{1/2}\phi) = b(\psi,\phi), \qquad \psi,\phi \in \mathcal{X}. \tag{3.1}$$

3.1. **Solution formulas.** By [10], the equation (3.1) has a unique solution $X : \mathcal{X} \to \mathcal{X}$, which admits the representation

$$(\psi, X\phi) = \int_0^\infty b\left(\exp(At)|A|^{1/2}\psi, \exp(At)|A|^{1/2}\phi\right) \, \mathrm{d}t, \qquad \psi,\phi \in \mathcal{X}. \tag{3.2}$$

The operator $X : \mathcal{X} \to \mathcal{X}$ can be uniquely extended to a bounded operator $X : \mathcal{H} \to \mathcal{H}$, since the solution of the operator Lyapunov equation is unique and $\mathcal{X}$ is assumed to be dense in $\mathcal{H}$. However, the formula (3.2) only holds for $\psi, \phi \in \mathcal{X}$.

Since A is assumed to have a compact resolvent, there are orthonormal eigenvectors $\{\psi_i\}_{i\in\mathbb{N}}$ associated with the eigenvalues $\lambda_i < 0$ of A that span the whole space $\mathcal{H}$. It follows that the solution $X : \mathcal{H} \to \mathcal{H}$ of (3.1) is equivalently defined by the relation

$$(\psi_i, X\psi_j) = -b(\psi_i, \psi_j)\frac{\sqrt{\lambda_i\lambda_j}}{\lambda_i + \lambda_j}. \tag{3.3}$$

3.2. **Low-rank approximation.** Motivated by techniques for the finite-dimensional case [9, 8, 13], we derive low-rank approximations for X from (3.3) via approximating the scalar function $1/z$ by a sum of exponentials. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) < 0$ such an approximation is obtained from numerical quadrature applied to the integral representation $-1/z = \int_0^\infty e^{tz}\,\mathrm{d}t$. Sinc quadrature [22] yields the following approximation, see [8, Lemma 5] and [13, Sec. 5.2].

Lemma 3.1. *Let $k \in \mathbb{N}$ and consider $z \in \mathbb{C}$ with $\mathrm{Re}(z) \leq -1$. Defining the quadrature nodes and weights*

$$t_p = \log\left(\exp(ph_{\mathrm{St}}) + \sqrt{1 + \exp(2ph_{\mathrm{St}})}\right), \quad \omega_p = h_{\mathrm{St}}/\sqrt{1 + \exp(2ph_{\mathrm{St}})}, \quad -k \leq p \leq k,$$

with $h_{\mathrm{St}} = \pi/\sqrt{k}$, yields the approximation error

$$\left|\int_0^\infty \exp(tz)\ dt - \sum_{p=-k}^{k} \omega_p \exp(t_p z)\right| \leq C_{\mathrm{St}} \exp(|\operatorname{Im}(z)|/\pi) \exp(-\pi\sqrt{k}). \tag{3.4}$$

The constant C_{St} is independent of z and k.

When restricting (3.4) to real z with $z \leq -1$, the constant C_{St} can be estimated numerically as $C_{\mathrm{St}} \approx 2.75$.

To utilize Lemma 3.1 for low-rank approximation, we assume that the linear bounded operator $|A|^{-1/2}B$ has finite rank. In particular, this implies that the Hilbert-Schmidt norm $\||A|^{-1/2}B\|_{\mathrm{HS}(\mathcal{U},\mathcal{H})}$ is finite.

Theorem 3.2. *Consider a self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$ with compact resolvent and eigenvalues $0 > \lambda_1 \geq \lambda_2 \geq \cdots$. If $|A|^{-1/2}B$ has finite rank $m \in \mathbb{N}$ then the solution of the operator Lyapunov equation (3.1) can be approximated for every $k \in \mathbb{N}$ by a linear operator X_k of rank at most $(2k + 1)m$ such that*

$$\||A|^{-1/2}(X - X_k)|A|^{-1/2}\|_{\mathrm{HS}(\mathcal{H},\mathcal{H})} \leq \frac{C_{\mathrm{St}}\||A|^{-1/2}B\|^2_{\mathrm{HS}(\mathcal{U},\mathcal{H})}}{2|\lambda_1|} \cdot \exp(-\pi\sqrt{k}).$$

Proof. For t_p and ω_p defined as in Lemma 3.1, we set

$$\tilde{t}_p = \frac{1}{2|\lambda_1|}t_p, \quad \tilde{\omega}_p = \frac{1}{2|\lambda_1|}\omega_p$$

for the linear operator defined by

$$(\psi, X_k\phi) = \sum_{p=-k}^{k} \tilde{\omega}_p b\left(\exp(\tilde{t}_p A)|A|^{1/2}\psi, \exp(\tilde{t}_p A)|A|^{1/2}\phi\right), \qquad \psi, \phi \in \mathcal{X}.$$

Being a sum of $2k + 1$ terms of rank at most m, X_k has rank at most $(2k + 1)m$.

For eigenvectors ψ_i, ψ_j it follows from (3.3) that

$$\frac{\left|(\psi_i, X\psi_j) - (\psi_i, X_k\psi_j)\right|}{\sqrt{\lambda_i\lambda_j}} = \Big|\frac{-1}{\lambda_i+\lambda_j} - \sum_{p=-k}^{k} \tilde{\omega}_p \exp(\tilde{t}_p(\lambda_i+\lambda_j))\Big|\big|b(\psi_i,\psi_j)\big|$$

$$= \frac{1}{2|\lambda_1|}\Big| -\frac{1}{z} - \sum_{p=-k}^{k} \omega_p \exp(t_p z)\Big|\big|b(\psi_i,\psi_j)\big|, \tag{3.5}$$

where $z = \frac{1}{2|\lambda_1|}(\lambda_i+\lambda_j)$. Applying the result of Lemma 3.1 thus yields

$$\begin{aligned}\||A|^{-1/2}(X-X_k)|A|^{-1/2}\|^2_{\mathrm{HS}(\mathcal{H},\mathcal{H})} &= \sum_{i,j=1}^{\infty} \frac{\left|(\psi_i, X\psi_j) - (\psi_i, X_k\psi_j)\right|^2}{\lambda_i\lambda_j} \\ &\le \frac{C_{\mathrm{St}}^2}{4|\lambda_1|^2}\exp(-\pi\sqrt{k})\sum_{i,j=1}^{\infty}\big|b(\psi_i,\psi_j)\big|^2.\end{aligned}$$

Noting that $\sum_{i,j=1}^{\infty}\big|b(\psi_i,\psi_j)\big|^2 = \||A|^{-1/2}B\|^4_{\mathrm{HS}(\mathcal{U},\mathcal{H})}$ concludes the proof. Q.E.D.

We now consider the eigenvalues of $|A|^{-1/2}X|A|^{-1/2}$, in the sense that (λ,ψ) with $\psi \in \mathcal{X}\setminus\{0\}$ is an eigenpair if

$$(\psi, X\phi) = \lambda(\psi,\phi)_{\mathcal{X}}, \qquad \phi \in \mathcal{X}.$$

Letting the jth largest such eigenvalue be denoted by $\lambda_j(|A|^{-1/2}X|A|^{-1/2})$, we have

$$\begin{aligned}\lambda_{(2k+1)m+1}(|A|^{-1/2}X|A|^{-1/2}) &\le \Big(\sum_{j=(2k+1)m+1}^{\infty} \lambda_j^2(|A|^{-1/2}X|A|^{-1/2})\Big)^{1/2} \\ &\le \||A|^{-1/2}(X-X_k)|A|^{-1/2}\|_{\mathrm{HS}(\mathcal{H},\mathcal{H})}\end{aligned}$$

by the Schmidt-Mirsky theorem. Hence, the result of Theorem 3.2 implies that these eigenvalues decay exponentially in $\sqrt{k}$.

Remark 3.3. The above results can be extended to general $\alpha \ge 0$, provided that $|A|^{-\alpha}B$ is bounded and has finite rank m. In this case we are studying the eigenvalues (λ,ψ) with $\psi \in \mathcal{H}_\alpha \setminus \{0\}$ defined variationally by $(\psi, X\phi) = \lambda(\psi,\phi)_{\mathcal{H}_\alpha}$, $\phi \in \mathcal{H}_\alpha$. A variation of the result of Theorem 3.2 yields

$$\begin{aligned}\lambda_{(2k+1)m+1}(|A|^{-\alpha}X|A|^{-\alpha}) &\le \||A|^{-\alpha}(X-X_k)|A|^{-\alpha}\|_{\mathrm{HS}(\mathcal{H},\mathcal{H})} \\ &\le \frac{C_{\mathrm{St}}\||A|^{-\alpha}B\|^2_{\mathrm{HS}(\mathcal{U},\mathcal{H})}}{2|\lambda_1|}\cdot\exp(-\pi\sqrt{k}).\end{aligned}$$

Analogous to (1.3), this implies

$$\lambda_k(|A|^{-\alpha}X|A|^{-\alpha}) \lesssim \gamma^{\sqrt{k}} \quad \text{with} \quad \gamma = \exp(-\pi/\sqrt{2m}).$$

In particular for $\alpha = 0$, this means that the eigenvalues of X, defined via

$$(\psi, X\phi) = \lambda(\psi,\phi), \qquad \phi \in \mathcal{H}.$$

decay exponentially in $\sqrt{k}$. This, however, requires that B itself is bounded and has finite rank. $\diamond$

4. Extensions to the non-self-adjoint case

In this section, we illustrate how the results of Section 3 can be extended to the more general setting described in Section 2. In particular, A is assumed to have a Riesz basis of eigenvectors $\{\psi_i\}_{i\in\mathbb{N}}$. This implies that there is a bounded operator $Q:\mathcal{H}\to\mathcal{H}$ with bounded inverse, such that

$$\psi_i = Q\hat{\psi}_i, \quad i\in\mathbb{N}, \tag{4.1}$$

for an orthonormal basis $\{\hat{\psi}_i\}_{i\in\mathbb{N}}$. The condition number $c_Q = \|Q^{-1}\|\|Q\|$, which becomes 1 for self-adjoint A, measures the non-normality of A.

Theorem 4.1. *Let $|A|^{-1}$ be compact and let A have a Riesz basis of eigenvectors, with the eigenvalues contained in the strip $\big[-\infty,-\delta\big)\times[-\theta\mathrm{i},\theta\mathrm{i}\big]\subset\mathbb{C}$ for some $\delta>0,\theta\geq 0$. If B has finite rank $m\in\mathbb{N}$ then the solution of the operator Lyapunov equation* (2.1) *can be approximated for every $k\in\mathbb{N}$ by a linear operator X_k of rank at most $(2k+1)m$ such that*

$$\|X-X_k\|_{\mathrm{HS}(\mathcal{H},\mathcal{H})} \leq \frac{c_Q^2 C_{\mathrm{St}}}{2\delta}\exp\left(\frac{\theta}{\delta\pi}\right)\exp(-\pi\sqrt{k})\|B\|^2_{\mathrm{HS}(\mathcal{U},\mathcal{H})}, \tag{4.2}$$

with c_Q defined as above and C_{St} from Lemma 3.1.

Proof. Under the given assumptions, there is a unique bounded, nonnegative, and self-adjoint solution $X:\mathcal{H}\to\mathcal{H}$ of (2.1), see Section 2. Let ψ_i,ψ_j be eigenvectors corresponding to the eigenvalues λ_i,λ_j of A. By [11, Theorem 1.5], we have

$$(\psi_i, X\psi_j) = -\frac{\big(B'\psi_i, B'\psi_j\big)_{\mathcal{U}}}{\lambda_i+\overline{\lambda_j}} = \frac{\mathfrak{b}(z_1,z_2)}{\lambda_i+\overline{\lambda_j}}.$$

Analogous to the construction in the proof of Theorem 3.2, we define X_k by the relation

$$(\psi_i, X_k\psi_j) := -\sum_{p=-k}^{k}\tilde{\omega}_p\mathfrak{b}\left(\exp(\tilde{t}_pA)\psi_i, \exp(\tilde{t}_pA)\psi_j\right),$$

with $\tilde{t}_p = t_p/(2\delta)$ and $\tilde{\omega}_p=\omega_p/(2\delta)$. Using (4.1) and Lemma 3.1, it thus follows that

$$\begin{aligned}|(Q\hat{\psi}_i, XQ\hat{\psi}_j)-(Q\hat{\psi}_i, X_kQ\hat{\psi}_j)| &= |(\psi_i,X\psi_j)-(\psi_i,X_k\psi_j)| \\ &= \frac{1}{2\delta}\Big|-\frac{1}{z}-\sum_{p=-k}^{k}\omega_p\exp(t_pz)\Big|\big|\mathfrak{b}(\psi_i,\psi_j)\big| \\ &\leq \frac{C_{\mathrm{St}}}{2\delta}\exp\left(\frac{\theta}{\delta\pi}\right)\exp(-\pi\sqrt{k})\big|\mathfrak{b}(Q\hat{\psi}_i,Q\hat{\psi}_j)\big|,\end{aligned}$$

where $z:=\frac{1}{2\delta}(\lambda_i+\overline{\lambda_j})\in\big[-\infty,-1\big)\times\big[-\theta/\delta\mathrm{i},\theta/\delta\mathrm{i}\big]$. Using

$$\|X-X_k\|_{\mathrm{HS}(\mathcal{H},\mathcal{H})}\leq\|Q^{-1}\|^2\|Q^*(X-X_k)Q\|_{\mathrm{HS}(\mathcal{H},\mathcal{H})}, \quad \|Q^*B\|_{\mathrm{HS}(\mathcal{U},\mathcal{H})}\leq\|Q\|\,\|B\|_{\mathrm{HS}(\mathcal{U},\mathcal{H})},$$

this completes the proof. Q.E.D.

Again, the result of Theorem 4.1 implies that the eigenvalues of X decay exponentially in $\sqrt{k}$.

Remark 4.2. For highly non-normal A, the value of c_Q becomes very large and the bound (4.2) may not be representative of the actual behavior. For the finite-dimensional case, Sabino [18, Sec. 3.1] has developed bounds based on pseudospectra and has given an example demonstrating that higher non-normality may actually lead to faster eigenvalue decay.

Several variations of Theorem 4.1 are possible. For simplicity, let us from now on additionally assume that A has real spectrum and compact resolvent. Then one possible variation of (4.2) is given by

$$\|A^{-1}(X - X_k)A^{-*}\|_{\mathrm{HS}(\mathcal{H},\mathcal{H})} \leq \frac{c_Q^2 C_{\mathrm{St}}}{2\delta} \exp(-\pi\sqrt{k})\|A^{-1}B\|_{\mathrm{HS}(\mathcal{U},\mathcal{H})} ,$$

where $\|A^{-1}B\|_{\mathrm{HS}(\mathcal{U},\mathcal{H})}$ is assumed to be finite.

Now, let us assume that $-A$ is elliptic and Kato's square root theorem applies [3]. By a slight abuse of notation, let $A^{1/2}$ denote the square root of $-A$. Then another variation of (4.2) is given by

$$\|A^{-1/2}(X - X_k)(A^*)^{-1/2}\|_{\mathrm{HS}(\mathcal{H},\mathcal{H})} \leq \frac{c_Q^2 C_{\mathrm{St}}}{2\delta} \exp(-\pi\sqrt{k})\|A^{-1/2}B\|_{\mathrm{HS}(\mathcal{U},\mathcal{H})} .$$

Often, this norm can be related to standard Sobolev space estimates. For example, when considering Example 2.2, we can use the norm equivalences

$$c_1\|\psi\|_1 \leq \|A^{1/2}\psi\| \leq C_1\|\psi\|_1$$
$$c_2\|\psi\|_1 \leq \|(A^*)^{1/2}\psi\| \leq C_2\|\psi\|_1$$

for every $\psi \in H_0^1(0,1) = \mathcal{H}_{1/2} = \mathcal{H}_{1/2}^d$, where $\|\cdot\|_1$ denotes the H^1 norm. See, e.g., [4] for other examples.

5. Model problems and numerical experiments

In this section we present two types of numerical experiments, illustrating the decay of the eigenvalues for increasingly refined discretizations of Lyapunov operator equations arising from 1D and 2D model problems.

5.1. **Numerical experiments in 1D.** Let us consider the following model problem in $\mathcal{H} = L^2(0,1)$:

$$z_t(t,x) = \kappa z_{xx}(t,x) + \delta(x - x_0)u(t), \qquad z(x,0) \equiv 0,$$
$$y(t) = \int_{1/3}^{2/3} z(t,\xi)\, d\xi, \qquad z(0,t) = z(1,t) = 0.$$

This model problem can be realized in $l^2(\mathbb{N})$ as a diagonal system. For the Dirichlet Laplace operator $A = \partial_{xx}$, defined in $H^2(0,1) \cap H_0^1(0,1)$, we have the spectral decomposition

$$A\phi_j = \lambda_j\phi_j, \quad \text{where} \quad \lambda_j = -\pi^2 j^2, \quad \phi_j(x) = \sqrt{2}\sin(j\pi x).$$

We use a Matlab implementation of the extended Krylov subspace method [21] to compute a highly accurate low-rank approximation to the solution of the Lyapunov matrix equation

$$A_n X_{B,n} + X_{B,n} A_n = -b_n b_n^*$$

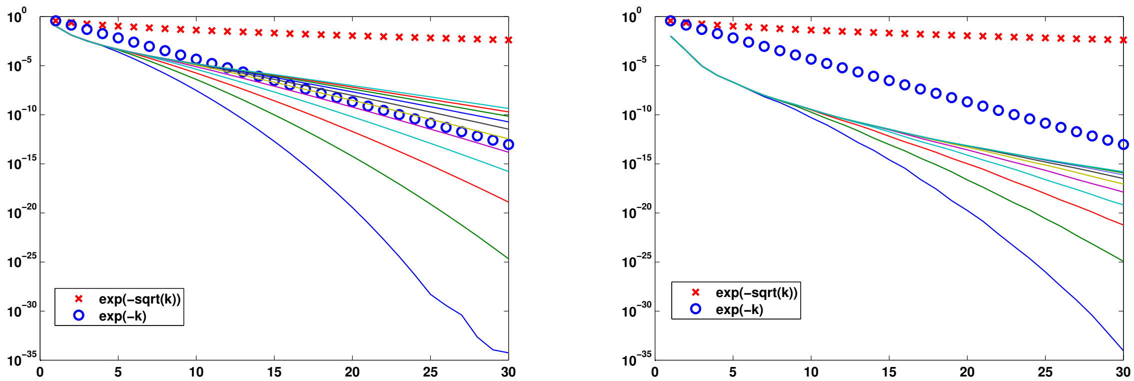
for each $n \in \{2^i \ : \ i = 5, \cdots, 16\}$, where $A_n = \mathsf{diag}(\lambda_j \ : \ j = 1, \cdots, n)$ and

$$b_n = (\langle \delta(x - 1/2), \phi_j\rangle_{H^{-1}\times H_0^1})_{j=1}^n = \left(\sqrt{2}\sin(j\pi/2)\right)_{j=1}^n.$$

This discretization corresponds to a Galerkin projection of the Lyapunov operator equation onto the subspace spanned by $\phi_1, \ldots, \phi_n$.

Analogously, we consider the Lyapunov matrix equation for the observability Gramian:

$$A_n X_{C,n} + X_{C,n} A_n = -c_n^* c_n,$$



(a) Controllability Gramian

(b) Observability Gramian

FIGURE 1. 1D model problem: Solid lines show the largest 30 eigenvalues of $X_{B,n}$ (left plot) and $X_{C,n}$ (right plot) for $n = 2^6$ (bottom line) to $n = 2^{16}$ (top line).

with the row vector

$$c_n = \left(-\frac{\sqrt{2}}{j\pi}\left(\cos\left(\frac{2\pi}{3}j\right) - \cos\left(\frac{\pi}{3}j\right)\right)\right)_{j=1}^{n}.$$

The eigenvalues of the computed Gramians $X_{B,n}$ and $X_{C,n}$ are shown in Figure 1. The eigenvalues appear to decay exponentially (or even superexponentially) for small n, as predicted by (1.3). However, as n increases, the decay appears to deteriorate and becomes subexponential. In fact, the "square root expotential decay" predicted by our results is clearly visible for large n.

Remark 5.1. The numerical results from Figure 1 appear to indicate that the eigenvalue decay rate deteriorates faster than what is indicated by the estimate (1.3). For example, even for moderate sizes of $\kappa(A)$ (e.g., for $M = 2^{11}$ we have $\kappa(A) = O(10^6)$) the decay observed in Figure 1(a) appears to be slower than predicted by the asymptotic estimate (1.3). On the other hand the decay appears to be much faster in Figure 1(b). This indicates that the structure of B influences the decay rate of eigenvalues. For some finite dimensional results in this direction, see [24, Section 2.1].

5.2. **Numerical experiments in 2D.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let γ_0 and γ_1 denote the Dirichlet and the Neumann trace on the spaces $H^1(\Omega)$ and $H^2(\Omega)$ [25, Chaper 13]. Assuming that the boundary of Ω decomposes into two disjoint nonempty sets Γ_0 and Γ_1 so that $\partial\Omega = \mathrm{cls}(\Gamma_0) \cup \mathrm{cls}(\Gamma_1)$, we define the mixed Dirichlet–Neumann space

$$H^1_{DN}(\Omega) = \{\phi \in H^1(\Omega) \ : \ \gamma_0^{\Gamma_0}\phi = 0\},$$

where $\gamma_0^{\Gamma_0}$ denotes the restriction of γ_0 onto Γ_0.

We consider the following control problem [25]:

$$\partial_t z(t,\xi) = \triangle z(t,\xi) \qquad \xi \in \Omega, \tag{5.1}$$

$$\nabla_\nu z(t,\xi) = b \qquad \xi \in \Gamma_1 \tag{5.2}$$

$$z(t,\xi) = 0 \qquad \xi \in \Gamma_2 \tag{5.3}$$

$$y(t) = \int_\Omega c(\xi) z(t,\xi)\ d\xi. \tag{5.4}$$

The formal expression ∇_ν should be understood as the action of the Neumann trace operator$\gamma_1^{\Gamma_1}$ restricted to Γ_1. The weak formulation of (5.1)–(5.3) reads: Seek $z(t,\cdot) \in H^1_{DN}(\Omega)$ such that

$$\int_\Omega \partial_t z\phi\ dx = \int_\Omega \nabla z \nabla \phi\ dx - \int_{\Gamma_1} b\gamma_0\phi\ dS, \qquad \phi \in H^1_{DN}(\Omega).$$

Let $|A|$ be the operator defined by $(|A|^{1/2}z, |A|^{1/2}\phi) = -\int_\Omega \nabla z \nabla\phi\ dx$ for $z,\phi \in H^1_{DN}(\Omega)$. The functional $B : b \mapsto b\int_{\Gamma_1} b\gamma_0\phi\ dS$ is a continuous functional on $H^1_{DN}(\Omega)$, that is, $|A|^{-1/2}B$ is bounded. This allows us to apply Theorem 3.2 to the corresponding Lyapunov operator equation for the controllability Gramian.

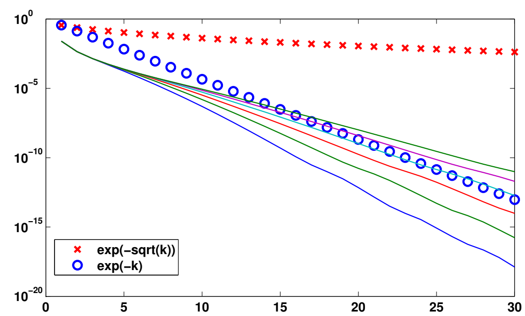


FIGURE 2. 2D model problem: Solid lines show the largest 30 eigenvalues of the controllability Gramian for $M = 2^6$ (bottom line) to $M = 2^{11}$ (top line).

5.2.1. *Spectral element method for the controllability Gramian.* Let us consider (5.1)–(5.4) for $\Omega = [0,1]^2$, where we impose the Neumann boundary condition on the edge $[0,1]\times\{0\}$ and the Dirichlet boundary condition along the other three edges. We chose the function $b \equiv 1$ to define the controllability operator B.

In this setting, the Laplace operator with mixed boundary conditions has the eigenfunctions

$$\psi_{k,p}(x,y) = \sin(k\pi x)\cos\Big(p\pi y + \frac{\pi}{2}y\Big), \quad k \in \mathbb{N} \text{ and } p \in \mathbb{N}\cup\{0\},$$

belonging to the eigenvalues

$$\lambda_{k,p} = \pi^2\Big(k^2 + \Big(p+\frac{1}{2}\Big)^2\Big), \quad k \in \mathbb{N} \text{ and } p \in \mathbb{N}\cup\{0\}.$$

The corresponding entry of the right-hand side is given by

$$\int_{\Gamma_1} \gamma_0\psi_{k,p}\ dS = \int_0^1 \sin(k\pi x)dx = -\frac{\cos(\pi k) - 1}{\pi k}.$$

As before, the problem is discretized by a Galerkin projection onto the subspace

$$\mathcal{V}_M = \big\{\psi_{k,p}\ :\ k \le M \ \text{ and } \ p \le M\big\}.$$

The obtained results are presented in Figure 2.

5.2.2. *Finite element approximations.* By the stability of Galerkin projections for self-adjoint operators, the results of Section 3 uniformly hold for any such discretization of the Lyapunov operator equation considered above. In fact, the only assumption on the subspace $\mathcal{V}_M$, on which the problem will be projected, is that $\mathcal{V}_M$ is contained in $\mathcal{X} = H^1_{DN}(\Omega)$. No further regularity restrictions need to be imposed for our eigenvalue decay to hold. In turn, they also apply to standard finite element spaces, e.g. the space of piecewise linear functions, instead of spectral element spaces. In fact, preliminary numerical experiments, which we performed with finite element approximation on quasi-uniform meshes, showed a behavior similar to the one observed for spectral elements. However, we envision that the proper setting for the application of these results will be in the context of adaptive finite element procedures.

6. Conclusion

In [14], Opmeer has proved that the eigenvalues of Gramians for analytic control systems decay superpolynomially. In this paper, we have provided additional insight into the qualitative and quantitative behavior of this decay by establishing "square root exponential" decay bounds, via an extension of existing results for the finite-dimensional case [8].

Our results are of practical importance in model order reduction [6] of certain infinite-dimensional linear systems [17, 19], where they allow to gain a priori insight into the size of the reduced order model to guarantee a prescribed truncation error. A typical example of such a system is the heat equation with boundary Neumann control and distributed observation, such as the one considered in Section 5. For an alternative construction of finite-rank approximations to such infinite-dimensional systems, see [15].

We envision that this research could also be helpful in the context of adaptive finite element schemes for model reduction, where the errors from the model order reduction and from the finite element approximation error need to be balanced.

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