Primary objectives:

- Convex optimization
- Ellipsoid method
- A polynomial algorithm for linear programming
Part 6
Convex Optimization
Reminder: Convex functions

Convex function

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex function, if domain of $f$ is convex and for each $x, y \in \text{dom}(f)$ and $0 \leq \lambda \leq 1$ one has

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Example

$\| \cdot \|$ (any norm) is a convex function, since $\| \alpha \cdot x \| = |\alpha| \cdot \|x\|$ and $\|x + y\| \leq \|x\| + \|y\|$. Thus $\| \lambda x + (1 - \lambda)y \| \leq \lambda \|x\| + (1 - \lambda)\|y\|$. 
Sublevel sets

Definition $C_\alpha$

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ convex and $\alpha \in \mathbb{R}$, $C_\alpha = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is $\alpha$-sublevel set of $f$.

Lemma 6.1

If $f$ is convex, then $C_\alpha$ is a convex set for each $\alpha \in \mathbb{R}$.

Epigraph

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ convex, $\text{epi}(f) = \{(x, t) : x \in \text{dom}(f), f(x) \leq t\}$ is epigraph of $f$.

Lemma 6.2

$f$ is convex if and only if $\text{epi}(f)$ is convex set.
A convex optimization problem is of the form

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq b_i \quad \text{for } i = 1, \ldots, m,
\end{align*}
\]

where \( f_i, i = 0, \ldots, m \) are convex functions.

**Example: Quadratic programming**

\( Q \in \mathbb{R}^{n \times n} \) positive semidefinite, \( c \in \mathbb{R}^n \), \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Convex quadratic program

\[
\begin{align*}
\min x^T Q x + c^T x \\
Ax & = b \\
x & \geq 0,
\end{align*}
\]

is convex optimization problem.
Binary search for minimum

- Suppose we can efficiently test whether convex set is empty or not.
- Search smallest $\beta \in \mathbb{R}$ such that convex set $C_\beta = \{ x \in \mathbb{R}^n : f_0(x) \leq \beta, f_1(x) \leq b_1, \ldots, f_m(x) \leq b_m \}$ is non-empty.
- Keep upper bound $U$ and lower bound $L$.
- **Test:** Whether $C_{(L+U)/2} = \emptyset$. If yes, then $L := (L + U)/2$. If no, then $U := (L + U)/2$.
- After $O(\log ((U - L)/\varepsilon))$ tests, one obtains a value of distance $\leq \varepsilon$ from the optimum value.
Theorem 6.3

If \( S \subseteq \mathbb{R}^n \) is closed and convex and \( x^* \notin S \), then there exists a hyperplane \( c^T x = \delta \) such that \( c^T s < \delta \) for each \( s \in S \) and \( c^T x^* > \delta \).
Balls and ellipsoids

The **unit ball** is the set $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. An **ellipsoid** $E(A, b)$ is the image of the unit ball under an affine map $t : \mathbb{R}^n \to \mathbb{R}^n$ with $t(x) = Ax + b$, where $A \in \mathbb{R}^{n \times n}$ is an invertible matrix and $b \in \mathbb{R}^n$ is a vector.

Clearly

$$E(A, b) = \{x \in \mathbb{R}^n \mid \|A^{-1} x - A^{-1} b\| \leq 1\}. \quad (13)$$

**Exercise**

*Consider the mapping $t(x) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}$. Draw the ellipsoid which is defined by $t$. What are the axes of the ellipsoid?*

**Volume of unit ball**

The **volume** of the unit ball is $V_n \sim \frac{1}{\pi^n} \left( \frac{2e\pi}{n} \right)^{n/2}$. The volume of ellipsoid $E(A, b)$ is equal to $|\det(A)| \cdot V_n$. 
Lemma 6.4 (Half-Ball Lemma)

The half-ball \( H = \{x \in \mathbb{R}^n \mid \|x\| \leq 1, x(1) \geq 0\} \) is contained in the ellipsoid

\[
E = \left\{ x \in \mathbb{R}^n \mid \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2 - 1}{n^2} \sum_{i=2}^{n} x(i)^2 \leq 1 \right\} \tag{14}
\]
Proof

Let \( x \) be contained in the unit ball, i.e., \( \|x\| \leq 1 \) and suppose further that \( 0 \leq x(1) \) holds. We need to show that

\[
\left( \frac{n+1}{n} \right)^2 \left( x(1) - \frac{1}{n+1} \right)^2 + \frac{n^2 - 1}{n^2} \sum_{i=2}^{n} x(i)^2 \leq 1 \tag{15}
\]

holds. Since \( \sum_{i=2}^{n} x(i)^2 \leq 1 - x(1)^2 \) holds we have

\[
\left( \frac{n+1}{n} \right)^2 \left( x(1) - \frac{1}{n+1} \right)^2 + \frac{n^2 - 1}{n^2} \sum_{i=2}^{n} x(i)^2
\leq \left( \frac{n+1}{n} \right)^2 \left( x(1) - \frac{1}{n+1} \right)^2 + \frac{n^2 - 1}{n^2} (1 - x(1)^2) \tag{16}
\]

This shows that (15) holds if \( x \) is contained in the half-ball and \( x(1) = 0 \) or \( x(1) = 1 \).
Proof cont.

Now consider the right-hand-side of (16) as a function of $x(1)$, i.e., consider

$$f(x(1)) = \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2 - 1}{n^2} (1 - x(1)^2). \quad (17)$$

The first derivative is

$$f'(x(1)) = 2 \cdot \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right) - 2 \cdot \frac{n^2 - 1}{n^2} x(1). \quad (18)$$

We have $f'(0) < 0$ and since both $f(0) = 1$ and $f(1) = 1$, we have $f(x(1)) \leq 1$ for all $0 \leq x(1) \leq 1$ and the assertion follows.
Corollary 6.5

The half-ball \( \{ x \in \mathbb{R}^n \mid x(1) \geq 0, \|x\| \leq 1 \} \) is contained in an ellipsoid \( E \),
whose volume is bounded by \( e^{-\frac{1}{2(n+1)}} \cdot V_n \).

Ellipsoids: Convenient notation

An ellipsoid \( \mathcal{E}(A, a) \) is the set
\[
\mathcal{E}(A, a) = \{ x \in \mathbb{R}^n \mid (x - a)^T A^{-1} (x - a) \leq 1 \},
\]
where \( A \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix and \( a \in \mathbb{R}^n \) is a vector.

Half-ellipsoid: \( \mathcal{E}(A, a) \cap (c^T x \leq c^T a) \) where \( c \in \mathbb{R}^n \)
Half-ellipsoid theorem

Proof of the correctness of next formula can be found in book of Grötschel, Lovász and Schrijver: *Geometric algorithms and combinatorial optimization*.

**Lemma 6.6 (Half-Ellipsoid-Theorem)**

The half-ellipsoid $E(A, b) \cap (c^T x \leq c^T a)$ is contained in the ellipsoid $E'(A', a')$ and one has $\text{vol}(E') / \text{vol}(E) \leq e^{-1/(2n)}$. Here $E'(A', a')$ is defined by

\begin{align*}
a' &= a - \frac{1}{n+1} b \quad \text{(19)} \\
A' &= \frac{n^2}{n^2 - 1} \left( A - \frac{2}{n+1} b b^T \right) \quad \text{(20)}
\end{align*}

where $b$ is the vector $b = A c / \sqrt{c^T A c}$.
Ellipsoid method

$S \subseteq \mathbb{R}^n$ convex compact set. Suppose the following:

I) We have an ellipsoid $\mathcal{E}_{\text{init}}$ which contains $S$.

II) We have separation oracle for $S$

Ellipsoid method decides whether $\text{vol}(S) < L$ or computes a point $x^* \in S$

Ellipsoid method

a) (Initialize): Set $\mathcal{E}(A, a) := \mathcal{E}_{\text{init}}$

b) If $\text{vol}(\mathcal{E}(A, a)) < L$, then stop.

c) If $a \in S$, then assert $S \neq \emptyset$ and stop

d) Otherwise, compute inequality $c^T x \leq \beta$ which is valid for $S$ and satisfies $c^T a > \beta$ and replace $\mathcal{E}(A, a)$ by $\mathcal{E}(A', a)$ computed with formula (19) and goto step c).
Theorem 6.7

The ellipsoid method computes a point in $S$ or asserts that $\text{vol}(S) < L$. The number of iterations is bounded by $2 \cdot n \ln(\text{vol}(E_{\text{init}})/L)$.

Further remarks

- The ellipsoid method can be used to solve convex programming problems in polynomial time under certain conditions. The exact formulation of the result involves some rounding arguments and is beyond the scope of a lecture on Optimization Methods in Finance. Instead we refer to the book of Grötschel, Lovász and Schrijver: *Geometric algorithms and combinatorial optimization* for a thorough account.

- The ellipsoid algorithm was in particular the first polynomial time method for linear programming.
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- Convex optimization ✔
- Ellipsoid method ✔
- A polynomial algorithm for linear programming