Primary objectives:

- Convex optimization
- Ellipsoid method
- A polynomial algorithm for linear programming

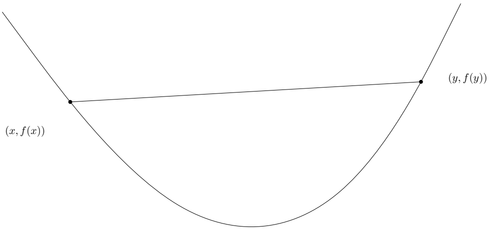
PART 6
CONVEX OPTIMIZATION

Reminder: Convex functions

Convex function

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex function, if domain of f is convex and for each $x, y \in \mathbf{dom}(f)$ and $0 \leqslant \lambda \leqslant 1$ one has

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$



Example

$\|\cdot\|$ (any norm) is a convex function, since $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ and $\|x+y\| \leqslant \|x\| + \|y\|$. Thus $\|\lambda x + (1-\lambda)y\| \leqslant \lambda \|x\| + (1-\lambda)\|y\|$.

Sublevel sets

Definition C_α

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ convex and $\alpha \in \mathbb{R}$, $C_\alpha = \{x \in \mathbb{R}^n : f(x) \leqslant \alpha\}$ is α-sublevel set of f.

Lemma 6.1

If f is convex, then C_α is a convex set for each $\alpha \in \mathbb{R}$.

Epigraph

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ convex, $\mathbf{epi}(f) = \{(x, t) : x \in \mathbf{dom}(f), f(x) \leqslant t\}$ is epigraph of f.

Lemma 6.2

f is convex if and only if $\mathbf{epi}(f)$ is convex set.

Convex optimization problem

Convex optimization problem

A convex optimization problem is of the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant b_i \quad \text{for } i = 1, \ldots, m, \end{array}$$

where f_i, $i = 0, \ldots, m$ are convex functions.

Example: Quadratic programming

$Q \in \mathbb{R}^{n \times n}$ positive semidefinite, $c \in \mathbb{R}^n$ $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Convex quadratic program

$$\begin{array}{c} \min x^T Qx + c^T x \\ \begin{array}{rcl} Ax & = & b \\ x & \geqslant & 0, \end{array} \end{array}$$

is convex optimization problem.

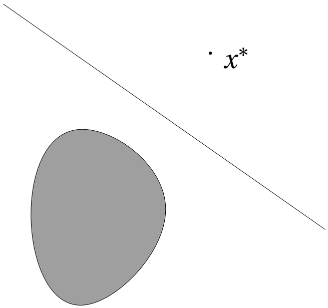
Binary search for minimum

- Suppose we can efficiently test whether convex set is empty or not
- Search smallest $\beta \in \mathbb{R}$ such that convex set $C_\beta = \{x \in \mathbb{R}^n \colon f_0(x) \leqslant \beta, f_1(x) \leqslant b_1, \ldots, f_m(x) \leqslant b_m\}$ is non-empty.
- Keep upper bound U and lower bound L
- **Test**: Whether $C_{(L+U)/2} = \emptyset$. If yes, then $L := (L+U)/2$. If no, then $U := (L+U)/2$.
- After $O(\log((U-L)/\varepsilon)$ tests, one obtains a value of distance $\leqslant \varepsilon$ from the optimum value.

Separating hyperplane

Theorem 6.3

If $S \subseteq \mathbb{R}^n$ is closed and convex and $x^* \notin S$, then there exists a hyperplane $c^T x = \delta$ such that $c^T s < \delta$ for each $s \in S$ and $c^T x^* > \delta$.



Balls and ellipsoids

The unit ball is the set $B = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$. An ellipsoid $E(A,b)$ is the image of the unit ball under an affine map $t: \mathbb{R}^n \to \mathbb{R}^n$ with $t(x) = Ax + b$, where $A \in \mathbb{R}^{n \times n}$ is an invertible matrix and $b \in \mathbb{R}^n$ is a vector.

Clearly

$$E(A,b) = \{x \in \mathbb{R}^n \mid \|A^{-1}x - A^{-1}b\| \leqslant 1\}. \tag{13}$$

Exercise

Consider the mapping $t(x) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x(1) \\ x(2) \end{pmatrix}$. *Draw the ellipsoid which is defined by* t. *What are the axes of the ellipsoid?*

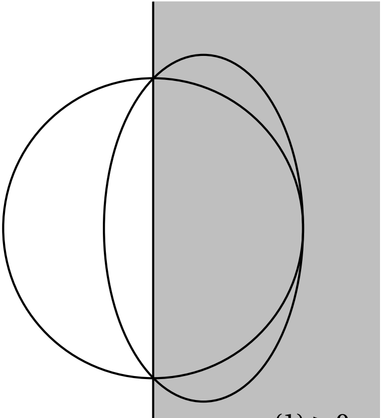
Volume of unit ball

The volume of the unit ball is $V_n \sim \frac{1}{\pi n} \left(\frac{2e\pi}{n}\right)^{n/2}$.
Volume of ellipsoid $E(A,b)$ is equal to $|\det(A)| \cdot V_n$.

Lemma 6.4 (Half-Ball Lemma)

The half-ball $H = \{x \in \mathbb{R}^n \mid \|x\| \leq 1, x(1) \geq 0\}$ *is contained in the ellipsoid*

$$E = \left\{ x \in \mathbb{R}^n \mid \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \leq 1 \right\} \tag{14}$$



Proof

Let x be contained in the unit ball, i.e., $\|x\| \leqslant 1$ and suppose further that $0 \leqslant x(1)$ holds. We need to show that

$$\left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \leqslant 1 \tag{15}$$

holds. Since $\sum_{i=2}^{n} x(i)^2 \leqslant 1 - x(1)^2$ holds we have

$$\begin{aligned}\left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 &+ \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \\ &\leqslant \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(1 - x(1)^2)\end{aligned} \tag{16}$$

This shows that (15) holds if x is contained in the half-ball and $x(1) = 0$ or $x(1) = 1$.

Proof cont.

Now consider the right-hand-side of (16) as a function of $x(1)$, i.e., consider

$$f(x(1)) = \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(1 - x(1)^2). \tag{17}$$

The first derivative is

$$f'(x(1)) = 2 \cdot \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right) - 2 \cdot \frac{n^2-1}{n^2} x(1). \tag{18}$$

We have $f'(0) < 0$ and since both $f(0) = 1$ and $f(1) = 1$, we have $f(x(1)) \leqslant 1$ for all $0 \leqslant x(1) \leqslant 1$ and the assertion follows.

Corollary 6.5

The half-ball $\{x \in \mathbb{R}^n \mid x(1) \geqslant 0, \|x\| \leqslant 1\}$ *is contained in an ellipsoid* E*, whose volume is bounded by* $e^{-\frac{1}{2(n+1)}} \cdot V_n$.

Ellipsoids: Convenient notation

An ellipsoid $\mathscr{E}(A, a)$ is the set
$\mathscr{E}(A, a) = \{x \in \mathbb{R}^n \mid (x-a)^T A^{-1}(x-a) \leqslant 1\}$, where $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix and $a \in \mathbb{R}^n$ is a vector.
Half-ellipsoid: $\mathscr{E}(A, a) \cap (c^T x \leqslant c^T a)$ where $c \in \mathbb{R}^n$

Half-ellipsoid theorem

Proof of the correctness of next formula can be found in book of Grötschel, Lovász and Schrijver: *Geometric algorithms and combinatorial optimization*.

Lemma 6.6 (Half-Ellipsoid-Theorem)

The half-ellipsoid $\mathscr{E}(A,b)\cap(c^T x \leqslant c^T a)$ *is contained in the ellipsoid* $\mathscr{E}'(A',d')$ *and one has* $\mathrm{vol}(\mathscr{E}')/\mathrm{vol}(\mathscr{E}) \leqslant e^{-1/(2n)}$.
Here $\mathscr{E}'(A',d')$ *is defined by*

$$d' = a - \frac{1}{n+1} b \tag{19}$$

$$A' = \frac{n^2}{n^2-1}\left(A - \frac{2}{n+1} b b^T\right), \tag{20}$$

where b *is the vector* $b = Ac/\sqrt{c^T A c}$.

Ellipsoid method

$S \subseteq R^n$ convex compact set. Suppose the following:

I) We have an ellipsoid $\mathscr{E}_{init}$ which contains S.

II) We have separation oracle for S

Ellipsoid method decides whether $\text{vol}(S) < L$ or computes a point $x^* \in S$

Ellipsoid method

a) (Initialize): Set $\mathscr{E}(A, a) := \mathscr{E}_{init}$

b) If $\text{vol}(\mathscr{E}(A, a)) < L$, then stop.

c) If $a \in S$, then assert $S \neq \emptyset$ and stop

d) Otherwise, compute inequality $c^T x \leqslant \beta$ which is valid for S and satisfies $c^T a > \beta$ and replace $\mathscr{E}(A, a)$ by $\mathscr{E}(A', a)$ computed with formula (19) and goto step c).

Theorem 6.7

The ellipsoid method computes a point in S or asserts that $\mathrm{vol}(S) < L$. *The number of iterations is bounded by* $2 \cdot n \ln(\mathrm{vol}(\mathscr{E}_{init})/L)$.

Further remarks

- The ellipsoid method can be used to solve convex programming problems in polynomial time under certain conditions. The exact formulation of the result involves some rounding arguments and is beyond the scope of a lecture on Optimization Methods in Finance. Instead we refer to the book of Grötschel, Lovász and Schrijver: *Geometric algorithms and combinatorial optimization* for a thorough account.
- The ellipsoid algorithm was in particular the first polynomial time method for linear programming.

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- Convex optimization ✔
- Ellipsoid method ✔
- A polynomial algorithm for linear programming