Primary objectives:

- Convex optimization ✔
- Ellipsoid method ✔
- A polynomial algorithm for linear programming

Part 7
The ellipsoid Method as a polynomial time algorithm for linear programming

Polynomial time algorithms

Algorithm runs in polynomial time if there exists a $k \in \mathbb{N}$ such that the number number of steps that it performs is bounded by $O(n^k)$, where n is the number of bits that are used to represent the input in a computer.

Sizes of numbers

- The size of an integer $i \in \mathbb{Z}$ is $\text{size}(i) = \lfloor \log(1 + |i|) \rfloor + 1$. It is an upper bound on the length of the bitstring, representing i.

Feasibility versus optimization

Feasibility problem

Given $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ defining a polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$, compute a point $x^* \in P$ or assert that P is empty.

Feasibility implies optimization

$x^* \in \mathbb{R}^n$ is an optimal solution of linear program $\max\{c^T x \colon x \in \mathbb{R}^n, Ax \leqslant b\}$ if and only if there exists $y^* \in \mathbb{R}^m$ such that $\binom{x^*}{y^*}$ is contained in the polyhedron $\{\binom{x}{y} \in \mathbb{R}^{n+m} \colon c^T x = b^T y, Ax \leqslant b, A^T y = c, y \geqslant 0\}$.

Bounded and full-dimensional polyhedra

Lemma 7.1

Suppose that $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ is full-dimensional and bounded with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. Let B be the largest absolute value of a component of A and b.

i) *The vertices of P are in the box $\{x \in \mathbb{R}^n \mid -n^{n/2}B^n \leqslant x \leqslant n^{n/2}B^n\}$. Thus P is contained in the ball around 0 with radius $n^n B^n$.*
ii) *The volume of P is bounded from below by $1/(n \cdot B)^{3n^2}$.*

Theorem 7.2

The ellipsoid method requires $O(n^3 \cdot \ln(n \cdot B))$ iterations to find a feasible point in a bounded and full-dimensioinal polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$, where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$ and B is an upper bound on the coefficients of A and b.

The general case

A has full column rank

Consider $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$, and suppose that A' is a maximal sub-matrix of A consisting of linearly independent columns. Clearly $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is nonempty, if and only if $P' = \{x \in \mathbb{R}^n : A'x \leqslant b\}$ is nonempty and for each $x^* \in P'$ one has that $(x^*, 0) \in P$. Therefore, w.l.o.g. assume that the matrix A is full-column rank.

Boundedness

Each vertex of $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is in box $\{x \in \mathbb{R}^n \mid -n^{n/2}B^n \leqslant x \leqslant n^{n/2}B^n\}$. Therefore, we can append the inequalities $-n^{n/2}B^n \leqslant x \leqslant n^{n/2}B^n$ to $Ax \leqslant b$ without changing the status of $P \neq \emptyset$ or $P = \emptyset$.

The general case

Exercise

Let $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ *be a polyhedron and* $\varepsilon > 0$ *be a real number. Show that* $P_\varepsilon = \{x \in \mathbb{R}^n \mid Ax \leqslant b + \varepsilon \cdot \mathbf{1}\}$ *is full-dimensional if* $P \neq \emptyset$.

Theorem 7.3

Farka's lemma The system $Ax \leqslant b$ *does not have a solution if and only if there exists a nonnegative vector* $\lambda \in \mathbb{R}^m_{\geqslant 0}$ *such that* $\lambda^T A = 0$ *and* $\lambda^T b = -1$.

Lemma 7.4

Let $\varepsilon = 1/\big((n+1)\cdot(n\cdot B)^n\big)$.
The system $Ax \leqslant b + \varepsilon\mathbf{1}$ *is infeasible if and only of* $Ax \leqslant b$ *is infeasible.*

Main conclusion

Theorem 7.5

The ellipsoid method can be used to decide whether a system of inequalities $Ax \leqslant b$ contains a feasible point, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. The number of iterations is bounded by a polynomial in n and $\log B$, where B is the largest absolute value of a coefficient of A and b.

Theorem 7.6

A linear program $\max\{c^T x \colon Ax \leqslant b\}$ can be solved in polynomial time in its binary encoding length.

Warning

We did not care about an important issue. The representation of the numbers in the intermediate steps of the algorithm (notice the square root in ellispoid formula) can be very large and need to be rounded to rational numbers with a polynomial encoding length. Further details are not very difficult but tedious.

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