Part 5

An algorithm for convex quadratic programming

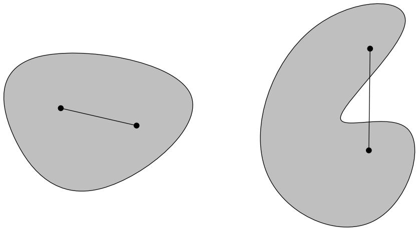
Primary objectives:

- Convex sets and functions
- Portfolio Optimization and convex minimization
- First oder condition
- Frank-Wolfe algorithm
- Analysis of Frank-Wolfe algorithm

Convex sets and functions

Convex set

A set $C \subseteq \mathbb{R}^n$ is convex if for each $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$, one has $\lambda x + (1 - \lambda) y \in C$.



Example

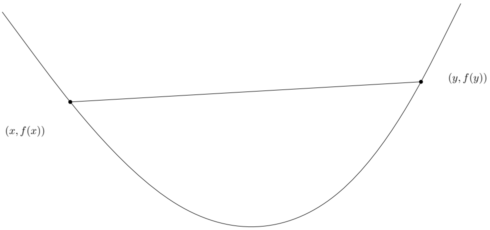
Set on the left is convex, on the right non-convex

Convex functions

Convex function

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex function, if domain of f is convex and for each $x, y \in \mathbf{dom}(f)$ and $0 \leqslant \lambda \leqslant 1$ one has

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$



Example

$\|\cdot\|$ (any norm) is a convex function, since $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ and $\|x+y\| \leqslant \|x\| + \|y\|$. Thus $\|\lambda x + (1-\lambda)y\| \leqslant \lambda\|x\| + (1-\lambda)\|y\|$.

Reminder

A symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is called positive semi-definite if $x^T Qx \geqslant 0$ for each $x \in \mathbb{R}^n$.

Theorem 5.1

Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric matrix. The following are equivalent.

i) *Q is positive definite.*

ii) *All Eigenvalues of Q real and non-negative*

iii) *$Q = U^T diag(\lambda_1, \ldots, \lambda_n) U$, where $U \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and $\lambda_i \in \mathbb{R}_{\geqslant 0}$ for $i = 1, \ldots, n$.*

Lemma 5.2

Let $Q \in \mathbb{R}^{n\times n}$ be symmetric and positive semidefinite, then $f(x) = x^T Qx$ is convex.

Portfolio optimization is problem of the form

$$\min f(x)$$
$$x \in P,$$

where $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is polytope (bounded polyhedron) and $f : \mathbb{R}^n \to \mathbb{R}$ convex and differentiable.

First order condition

Theorem (First oder condition)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and $\mathbf{dom}(f) \subseteq \mathbb{R}^n$ be convex, then f is convex if and only if for each $x, y \in \mathbf{dom}(f)$:
$f(y) \geqslant f(x) + \nabla f(x)^T (y - x)$.

Improving the solution

Consider

$$\min f(x) \quad x \in K \tag{12}$$

with $f : \mathbb{R}^n \to \mathbb{R}$ convex and differentiable and $K \subseteq \mathbf{dom}(f)$ convex.

If $x^* \in K$ is optimal solution of

$$\min \nabla f(x^*)^T x \quad x \in K$$

then x^* is optimal solution of (12).
If there exists $y^* \in K$ with $\nabla f(x^*)^T (y^* - x^*) < 0$, then x^* is not optimal and we can find better feasible solution on line-segment $\text{conv}(x^*, y^*)$.

The Frank-Wolfe algorithm

Input: $f : \mathbb{R}^n \to \mathbb{R}$ differentiable and polytope $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$
Task: Compute optimal solution of $\min_{x \in \mathrm{conv}\{x_1,\dots,x_k\}} f(x)$
Initialize: $x^{(0)}$ with arbitrary point in P
The algorithm iteratively computes $x^{(k+1)}$ from $x^{(k)}$.
If $\nabla f(x^{(k)}) = 0$, then $x^{(k)}$ is optimal and the algorithm ends.
Otherwise $x^{(k+1)}$ is computed as follows

- Compute optimal solution $y^{(k)}$ of linear program $\min\{\nabla f(x^{(k)})^T x : Ax \leqslant b\}$.
- Find $\lambda^* \in]0,1]$ minimizing $f(x^{(k)} + \lambda(y^{(k)} - x^{(k)})$.
- $x^{(k+1)} = x^{(k)} + \lambda^*(y^{(k)} - x^{(k)})$

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