Primary objectives:

- Convex sets and functions ✔
- Portfolio Optimization and convex minimization ✔
- First oder condition ✔
- Frank-Wolfe algorithm ✔
- Analysis of Frank-Wolfe algorithm

Analysis of Frank-Wolfe algorithm

We consider the case, where $f(x) = x^T Qx$ with $Q \in \mathbb{R}^{n \times n}$ positive semidefinite.

Primal (P)

$$\min f(x)$$
$$x \in P$$

Dual (D)

$$\max w(x)$$
$$x \in \mathbb{R}^n$$

where $w(x) = \min_{v \in P} \nabla f(x)^T (v - x) + f(x)$.

Weak and strong duality

Lemma: (Weak duality)

For any $x^* \in P$ and $y^* \in \mathbb{R}^n$ one has

$$f(x^*) \geqslant w(y^*),$$

i.e. $(P) \geqslant (D)$.

Lemma: (strong duality)

There exist $x^* \in P$ and $y^* \in \mathbb{R}^n$ with $f(x^*) = w(y^*)$.

Recall Taylor's theorem with Lagrange remainder

Let $g : \mathbb{R} \to \mathbb{R}$ be n-times continuously differentiable on $[a, x]$ and $n+1$-times differentiable on (a, x), then

$$g(x) = g(a) + \frac{g'(a)}{1!}(x-a) + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where $R_n(x) = \frac{g^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1}$ for some $\zeta \in (a, b)$.

Iteration of F-W

$x^{(k)}$ current sol., v opt sol of $\min\{\nabla f(x^{(k)})^T x \colon x \in P\}$,
$x^{(k+1)} = x^{(k)} + \lambda(v - x^{(k)})$, where $\lambda \in [0, 1]$, s.t., $f(x^{(k)} + \lambda(v - x^{(k)})$ is minimized.

Define $C = \max\{(x-y)^T Q(x-y) : x, y \in P\}$.

Lemma

$f(x^{(k+1)}) \leqslant f(x^{(k)}) - \frac{(w(x^{(k)} - f(x^{(k)}))^2}{4C}$.

Define

$g(x) = \frac{f(x) - w(x)}{4C}$, $h(x) = \frac{f(x) - f(x^*)}{4C}$, where x^* opt. sol of primal.

Theorem

$h(x^{(k)}) \leqslant 1/(k+3)$, i.e. the F.-W. algorithm converges.

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