Summary of last lecture

Primary objectives:

- Arbitrage: Definition and example ✔
- Duality and complementary slackness ✔
- Fundamental theorem of asset pricing ✔
- Arbitrage detection using linear programming
Scenario

- Portfolio of derivative securities (European call options) $S^i$, $i = 1, \ldots, n$ of one security $S$ is determined by vector $(x_1, \ldots, x_n)$
- Payoff of portfolio is $\Psi^x(S_1) = \sum_{i=1}^n \Psi_i(S_1)x_i$, where $\Psi_i(S_1) = \max\{S_1 - K_i, 0\}$, where $K_i$ is strike price $K_i$ (piecewise linear function with one breakpoint!)
- Cost of performing portfolio at time 0:
  $$\sum_{i=1}^n S^i_0 x_i.$$  

Determine arbitrage possibility

- Negative cost of portfolio with nonnegative payoff (type A)
- Cost zero and positive payoff (type B)
Observation

Nonnegative payoff

Payoff is piecewise linear in $S_1$ with breakpoints $K_1, \ldots, K_n$. Payoff is nonnegative on $[0, \infty)$, if nonnegative at 0 and at all breakpoints and right-derivative at $K_n$ is nonnegative (assume $K_1 < K_2 < \cdots < K_n$).

Formally:

\[
\Psi^x(0) \geq 0 \\
\Psi^x(K_j) \geq 0, \ j = 1, \ldots, n \\
\Psi^x(K_n + 1) - \Psi^x(K_n) \geq 0.
\]
Linear program

\[
\begin{align*}
\text{min} & \sum_{i=1}^{n} S_0^i x_i \\
\sum_{i=1}^{n} \Psi_i(0) x_i & \geq 0 \\
\sum_{i=1}^{n} \Psi_i(K_j) x_i & \geq 0, \ j = 1, \ldots, n \\
\sum_{i=1}^{n} (\Psi_i(K_n + 1) - \Psi_i(K_n)) x_i & \geq 0.
\end{align*}
\]
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<th>Proposition</th>
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<td>There is no type A arbitrage if and only if optimal objective value of LP is at least 0</td>
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<td>Suppose that there is no type A arbitrage. Then, there is no type B arbitrage if and only if the dual of LP has strictly feasible solution.</td>
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Constraint matrix

- \[ \Psi_i(K_j) = \max\{K_j - K_i, 0\} \]
- Constraint matrix \( A \) of LP has the form

\[
A = \begin{pmatrix}
K_2 - K_1 & 0 & 0 & \cdots & 0 \\
K_3 - K_1 & K_3 - K_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
K_n - K_1 & K_n - K_2 & K_n - K_3 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}
\]

Theorem

Let \( K_1 < K_2 < \cdots < K_n \) denote strike prices of European call options on the same underlying security with same maturity. There are no arbitrage opportunities if and only if prices \( S_0 \) satisfy

1. \( S_0^i > 0, \ i = 1, \ldots, n \)
2. \( S_0^i > S_0^{i+1}, \ i = 1, \ldots, n - 1 \)
3. \( C(K_i) := S_0^i \) defined on \( \{K_1, \ldots, K_n\} \) is strictly convex function
Summary of lectures on asset pricing

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