# Summary of last lecture

**Primary objectives:**

- Arbitrage: Definition and example ✔
- Duality and complementary slackness ✔
- Fundamental theorem of asset pricing ✔
- Arbitrage detection using linear programming

# Scenario

- Portfolio of derivate securities (European call options) $S^i$, $i = 1, \ldots, n$ of one security $S$ is determined by vector $(x_1, \ldots, x_n)$
- Payoff of portfolio is $\Psi^x(S_1) = \sum_{i=1}^n \Psi_i(S_1)x_i$, where $\Psi_i(S_1) = \max\{(S_1 - K_i), 0\}$, where $K_i$ is strike price $K_i$ (piecewise linear function with one breakpoint!)
- Cost of performing portfolio at time 0:

$$\sum_{i=1}^n S_0^i x_i.$$

**Determine arbitrage possibility**

- Negative cost of portfolio with nonnegative payoff (type A)
- Cost zero and positive payoff (type B)

# Observation

## Nonnegative payoff

Payoff is piecewise linear in $S_1$ with breakpoints $K_1, \ldots, K_n$.
Payoff is nonnegative on $[0, \infty)$, if nonnegative at 0 and at all breakpoints and right-derivative at $K_n$ is nonnegative (assume $K_1 < K_2 < \cdots < K_n$).
Formally:

$$\Psi^x(0) \geqslant 0$$
$$\Psi^x(K_j) \geqslant 0, j = 1, \ldots, n$$
$$\Psi^x(K_n + 1) - \Psi^x(K_n) \geqslant 0.$$

# Linear program

$$
\begin{aligned}
\min \textstyle\sum_{i=1}^{n} S_0^i x_i & & \\
\textstyle\sum_{i=1}^{n} \Psi_i(0) x_i & \geqslant & 0 \\
\textstyle\sum_{i=1}^{n} \Psi_i(K_j) x_i & \geqslant & 0, j = 1, \ldots, n \\
\textstyle\sum_{i=1}^{n} \left(\Psi_i(K_n + 1) - \Psi_i(K_n)\right) x_i & \geqslant & 0.
\end{aligned}
$$

**Proposition**

There is no type A arbitrage if and only if optimal objective value of LP is at least 0

**Proposition**

Suppose that there is no type A arbitrage. Then, there is no type B arbitrage if and only if the dual of LP has strictly feasible solution.

# Constraint matrix

- $\Psi_i(K_j) = \max\{K_j - K_i, 0\}$
- Constraint matrix $A$ of LP has the form

$$A = \begin{pmatrix} K_2 - K_1 & 0 & 0 & \cdots & 0 \\ K_3 - K_1 & K_3 - K_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_n - K_1 & K_n - K_2 & K_n - K_3 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

**Theorem**

Let $K_1 < K_2 < \cdots < K_n$ denote strike prices of European call options on the same underlying security with same maturity. There are no arbitrage opportunities if and only if prices $S_0^i$ satisfy

1. $S_0^i > 0$, $i = 1, \ldots, n$
2. $S_o^i > S_0^{i+1}$, $i = 1, \ldots, n-1$
3. $C(K_i) := S_0^i$ defined on $\{K_1, \ldots, K_n\}$ is strictly convex function

# Summary of lectures on asset pricing

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- Arbitrage: Definition and example ✔
- Duality and complementary slackness ✔
- Fundamental theorem of asset pricing ✔
- Arbitrage detection using linear programming ✔