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Computer Algebra

Fall 2010

Assignment Sheet 1

Exercises marked with a $\star$ can be handed in for bonus points. Due date is March 9.

Exercise 1
Sort the following functions according to their asymptotic growth. Indicate which pairs of functions satisfy $f = O(g)$, $f = \Omega(g)$, and $f = \Theta(g)$. Show that your answer is correct.

$$2^{5+\log n},\ \sqrt{n},\ 3^n,\ 17,\ \log n^2,\ 2^{\log^2 n},\ \log n,\ e^{\log n},\ 15n,\ n^7+3n^2,\ 2^{5\log n},\ 2^n,\ \log^2 n$$

Note: $\log n$ without an indicated base is always base 2.

Exercise 2 ($\star$)
Let $f, g : \mathbb{N} \to \mathbb{R}_+$. Show that $f = O(g)$ if and only if $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$.

Exercise 3
We can write the recursion that defines Fibonacci numbers as

$$\begin{pmatrix} F_k \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{k-1} \\ F_k \end{pmatrix}$$

Denoting the matrix by A, this implies that

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

1. Derive an explicit formula for the n-th Fibonacci number F_n. (*Hint*: Consider the eigenvalues and corresponding eigenvectors of A.)
2. Show that the bit length of F_n is $\Theta(n)$.

Evaluating the explicit formula directly is difficult because it involves irrational numbers.

3. Show how to compute A^n using $O(\log n)$ multiplications of 2×2 matrices.
4. Does this lead to a faster algorithm for computing Fibonacci numbers? (Consider also the time it takes to add and multiply numbers.)

Exercise 4 (⋆)
Implement the simple approach to multiplication, i.e. long multiplication or peasant's multiplication, in the framework provided in the `computer-algebra-2010` repository. Make sure that the `test_multiply` test cases pass and that your algorithm runs in time $O(n^2)$.

Read the README file that is available on https://svn.epfl.ch/svn/computer-algebra-2010/trunk/README. You have to add yourself to the group `computer-algebra-2010` on http://groups.epfl.ch/ to be able to access this file. It contains guidelines on how to obtain and build the source code and how to submit your solution.