Prof. Friedrich Eisenbrand
Nicolai Hähnle

Discussions from: May 11, 2010

Computer Algebra

Spring 2010

Assignment Sheet 6

Exercises marked with a $\star$ can be handed in for bonus points. Due date is May 25. The letter k denotes an arbitrary field.

Exercise 1 ($\star$)
Let $a = (a_i)_{i\in\mathbb{N}} \in k^{\mathbb{N}}$ be a linearly recurrent sequence of recursion order at most n and let $f = \sum_{j=0}^{d} f_j x^j \in k[x]$. Show that f is a characteristic polynomial of a if and only if $\sum_{j=0}^{d} f_j a_{i+j} = 0$ for all $0 \le i \le n-1$.

Exercise 2
Let $a = (a_i)_{i\in\mathbb{N}} \in k^{\mathbb{N}}$ be a linearly recurrent sequence and let $h = \sum_{i\in\mathbb{N}} a_i x^i \in k[[x]]$ be the corresponding formal power series. Let $f = f_0 + f_1 x + \cdots + f_d x^d \in k[x]$ (not necessarily $f_d \neq 0$), and let $r = f_0 x^d + f_1 x^{d-1} + \cdots + f_d$ be a reversal of f. Show that the following are equivalent:

1. f is a characteristic polynomial of a.
2. rh is a polynomial of degree strictly less than d.
3. a has recursion order at most n for some $n \ge d$, and $rh \equiv g \mod x^{2n}$, where g is a polynomial of degree strictly less than d.

Exercise 3
Let $a = (a_i)_{i\in\mathbb{N}} \in k^{\mathbb{N}}$ be a linearly recurrent sequence of recursion order at most n and define the polynomial $h = \sum_{j=0}^{2n-1} a_j x^j$. Show that the degree of $\gcd(h, x^{2n})$ is strictly less than n.

Exercise 4 ($\star$)
Show how to compute a characteristic polynomial of a linearly recurrent sequence $a \in k^{\mathbb{N}}$ of recursion order at most n using the Extended Euclidean algorithm for polynomials given the first $2n$ terms $a_0, \ldots, a_{2n-1}$.

Note: By improving the analysis of Exercise 2, this approach can even be used to compute the *minimal* polynomial of a.