Exercises marked with a ⋆ can be handed in for bonus points. Due date is May 25. The letter $k$ denotes an arbitrary field.

**Exercise 1 (⋆)**
Let $a = (a_i)_{i \in \mathbb{N}} \in k^{\mathbb{N}}$ be a linearly recurrent sequence of recursion order at most $n$ and let $f = \sum_{j=0}^{d} f_j x^j \in k[x]$. Show that $f$ is a characteristic polynomial of $a$ if and only if $\sum_{j=0}^{d} f_j a_{i+j} = 0$ for all $0 \leq i \leq n-1$.

**Exercise 2**
Let $a = (a_i)_{i \in \mathbb{N}} \in k^{\mathbb{N}}$ be a linearly recurrent sequence and let $h = \sum_{i \in \mathbb{N}} a_i x^i \in k[[x]]$ be the corresponding formal power series. Let $f = f_0 x^d + f_1 x^{d-1} + \cdots + f_d \in k[x]$ (not necessarily $f_d \neq 0$), and let $r = f_0 x^d + f_1 x^{d-1} + \cdots + f_d$ be a reversal of $f$. Show that the following are equivalent:

1. $f$ is a characteristic polynomial of $a$.
2. $rh$ is a polynomial of degree strictly less than $d$.
3. $a$ has recursion order at most $n$ for some $n \geq d$, and $rh \equiv g \mod x^{2n}$, where $g$ is a polynomial of degree strictly less than $d$.

**Exercise 3**
Let $a = (a_i)_{i \in \mathbb{N}} \in k^{\mathbb{N}}$ be a linearly recurrent sequence of recursion order at most $n$ and define the polynomial $h = \sum_{j=0}^{2n-1} a_j x^j$. Show that the degree of $\text{gcd}(h, x^{2n})$ is strictly less than $n$.

**Exercise 4 (⋆)**
Show how to compute a characteristic polynomial of a linearly recurrent sequence $a \in k^{\mathbb{N}}$ of recursion order at most $n$ using the Extended Euclidean algorithm for polynomials given the first $2n$ terms $a_0, \ldots, a_{2n-1}$.

*Note:* By improving the analysis of Exercise 2, this approach can even be used to compute the minimal polynomial of $a$. 

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Assignment Sheet 6