Operads and their bimodules

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Slogan

Operads

- parametrize n-ary operations, and
- govern the identities that they must satisfy.

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Bimodules over operads

- parametrize higher, “up to homotopy” structure on homomorphisms.

Monoids and modules

Monoids of sets

A *monoid* consists of

- a set A and
- a function $\mu : A \times A \to A : (a, b) \mapsto a \cdot b$ called *multiplication*

such that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in A$.

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such that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in A$.

I.e.,

$$\begin{array}{ccc} A \times A \times A & \xrightarrow{\mu \times Id_A} & A \times A \\ {\scriptstyle Id_A \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \times A & \xrightarrow{\mu} & A \end{array}$$

commutes.

Examples of monoids

- The set $\mathbb{R}$ of real numbers, where

$$\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

  is the usual multiplication of real numbers.
- The set $\mathcal{M}_n(\mathbb{R})$ of $(n \times n)$-matrices with real coefficients, where

$$\mu : \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$$

  is ordinary matrix multiplication.
- Any group is a monoid.

Homomorphisms of monoids

Let (A, μ) and (A', μ') be monoids.

A *homomorphism* from (A, μ) to (A', μ') is a function $f : A \to A'$ such that $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in A$.

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I.e.,

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & A' \times A' \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \mu'} \\ A & \xrightarrow{f} & A' \end{array}$$

commutes.

Example of a homomorphism

The determinant function

$$\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$$

is a homomorphism of monoids, with respect to the usual multiplication operations:

$$\det(AB) = \det(A) \cdot \det(B).$$

Commutative monoids of sets

A monoid (A, μ) is *commutative* if

$$a \cdot b = b \cdot a$$

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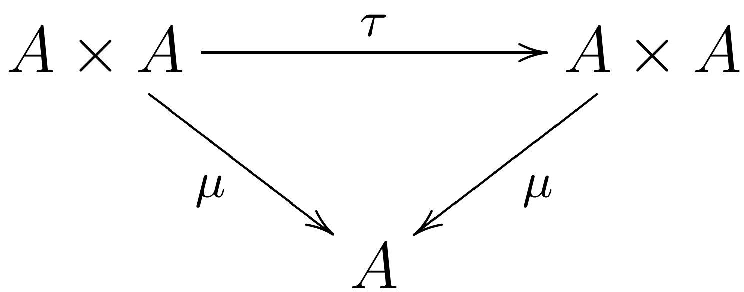

commutes, where

$$\tau(a, b) = (b, a)$$

for all $a, b \in A$.

Examples of commutative monoids

- The set $\mathbb{R}$ with its usual multiplication is a commutative monoid.
- The set $\mathbb{N}$ of natural numbers, where

$$\mu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

  is the usual addition of natural numbers, is a commutative monoid.
- Any abelian group is a commutative monoid.

Left modules over monoids

Let (A, μ) be a monoid.

A *left A-module* consists of a set M, together with a function

$$\lambda : A \times M \longrightarrow M : (a, x) \mapsto a * x$$

such that $(a \cdot b) * x = a * (b * x)$ for all $a, b \in A$ and $x \in M$.

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$$\begin{array}{ccc} A \times A \times M & \xrightarrow{Id_A \times \lambda} & A \times M \\ {\scriptstyle \mu \times Id_M}\downarrow & & \downarrow{\scriptstyle \lambda} \\ A \times M & \xrightarrow{\lambda} & M \end{array}$$

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Right modules over monoids

Let (A, μ) be a monoid.

A *right* A*-module* consists of a set M, together with a function

$$\rho : M \times A \longrightarrow M : (x, a) \mapsto x * a$$

such that $x * (a \cdot b) = (x * a) * b$ for all $a, b \in A$ and $x \in M$.

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$$\begin{array}{ccc} M \times A \times A & \xrightarrow{\rho \times Id_A} & M \times A \\ {\scriptstyle Id_M \times \mu} \downarrow & & \downarrow {\scriptstyle \rho} \\ M \times A & \xrightarrow{\rho} & M \end{array}$$

commutes.

Bimodules over monoids

Let (A, μ) be a monoid. Let (M, λ) and (M, ρ) be left and right A-modules.

Then (M, λ, ρ) is an A-*bimodule* if $(a * x) * b = a * (x * b)$ for all $a, b \in A$ and $x \in M$.

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$$\begin{array}{ccc} A \times M \times A & \xrightarrow{\lambda \times Id_A} & M \times A \\ {\scriptstyle Id_A \times \rho} \downarrow & & \downarrow {\scriptstyle \rho} \\ A \times M & \xrightarrow{\lambda} & M \end{array}$$

commutes.

Examples of modules

- Let
$$\lambda : \mathcal{M}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n : (A, \vec{v}) \mapsto A\vec{v}$$
denote the usual multiplication of a column vector by a matrix. For all $A, B \in \mathcal{M}_n(\mathbb{R})$ and for all $\vec{v} \in \mathbb{R}^n$,
$$A(B\vec{v}) = (AB)\vec{v}.$$
- Let
$$\lambda : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+ : (n, r) \mapsto n + r$$
denote the usual addtion operation. For all $m, n \in \mathbb{N}$ and for all $r \in R$,
$$m + (n + r) = (m + n) + r.$$

Generating n-ary operations

A binary operation

$$\mu : X \times X \to X : (x, y) \mapsto x \cdot y$$

on a set X gives rise to numerous n-ary operations, e.g.,

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$$X \times X \times X \to X : (x, y, z) \mapsto x \cdot (y \cdot z),$$

$$X \times X \times X \times X \to X : (w, x, y, z) \mapsto w \cdot ((z \cdot x) \cdot y),$$

$$X \times X \times X \times X \times X \to X : (v, w, x, y, z) \mapsto ((x \cdot v) \cdot z) \cdot (y \cdot w),$$

Etc.!

Generating n-ary operations

Need to organize and systematize all this information, i.e., to

- enumerate in a reasonable way all possible n-ary operations;
- encode relations among various n-ary operations.

A helpful descriptive tool

Planar trees labeled with permutations are useful for encoding n-ary operations.

Operads of sets

Symmetric sequences of sets

A *symmetric sequence* consists of a family of sets

$$\mathcal{X} = \{\mathcal{X}(n)\}_{n\geq 0}$$

where each $\mathcal{X}(n)$ is endowed with a right action of the symmetric (permutation) group $\mathfrak{S}_n$,

$$\alpha_n : \mathcal{X}(n) \times \mathfrak{S}_n \to \mathcal{X}(n) : (x, \sigma) \mapsto x * \sigma,$$

i.e., $(x * \sigma) * \sigma' = x * (\sigma\sigma')$ for all $x \in X$ and $\sigma, \sigma' \in \mathfrak{S}_n$.

Operads of sets

An *operad* consists of a symmetric sequence

$$\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0},$$

together with an "appropriately equivariant and associative" family of functions

$$\gamma_{n,\vec{m}} : \mathcal{P}(n) \times \Big(\mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n)\Big) \to \mathcal{P}\big(\sum_{i=1}^{n} m_i\big)$$

for all $n \in \mathbb{N}$ and all $\vec{m} = (m_1, ..., m_n) \in \mathbb{N}^n$.

Examples of operads

The *endomorphism operad* $\mathcal{E}_X$ on a set X.

For all $n \in \mathbb{N}$,

$$\mathcal{E}_X(n) = \{f : X^{\times n} \to X \mid f \text{ function}\},$$

on which $\mathfrak{S}_n$ acts by permuting inputs and

$$\gamma_{n,\vec{m}} : \mathcal{E}_X(n) \times \Big(\mathcal{E}_X(m_1) \times \cdots \times \mathcal{E}_X(m_n)\Big) \to \mathcal{E}_X\big(\sum_{i=1}^{n} m_i\big)$$

is given by composition of functions.

Examples of operads

The *commutative operad* $\mathcal{C}$.

For all $n \in \mathbb{N}$,

$$\mathcal{C}(n) = \{\star\},$$

endowed with the trivial action of $\mathfrak{S}_n$ and

$$\gamma_{n,\vec{m}} : \mathcal{C}(n) \times \Big(\mathcal{C}(m_1) \times \cdots \times \mathcal{C}(m_n)\Big) \to \mathcal{C}\big(\sum_{i=1}^{n} m_i\big)$$

is (essentially) the identity map.

Examples of operads

The *associative operad* $\mathcal{A}$.

For all $n \in \mathbb{N}$,

$$\mathcal{A}(n) = \mathfrak{S}_n,$$

on which $\mathfrak{S}_n$ acts by right multiplication and

$$\gamma_{n,\vec{m}} : \mathcal{A}(n) \times \Big(\mathcal{A}(m_1) \times \cdots \times \mathcal{A}(m_n)\Big) \to \mathcal{A}\big(\sum_{i=1}^{n} m_i\big)$$

is given by “block permutation.”

Algebras over operads I

Let $(\mathcal{P}, \gamma)$ be an operad.

A $\mathcal{P}$-*algebra* consists of a set X, together with an appropriately equivariant family of functions

$$\{\mu_n : \mathcal{P}(n) \times X^{\times n} \to X \mid n \geq 0\},$$

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i.e., for every element $p \in \mathcal{P}(n)$, there is an n-ary operation on X:

$$\mu_n(p, -) : X^{\times n} \to X.$$

Algebras over operads II

Furthermore, the following diagram must commute for all $n \in \mathbb{N}$ and all $\vec{m} = (m_1, ..., m_n) \in \mathbb{N}^n$.

$$
\begin{array}{ccc}
\mathcal{P}(n) \times \Big(\mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n)\Big) \times X^{\times m} & \xrightarrow{\gamma_{n,\vec{m}} \times Id} & \mathcal{P}(m) \times X^{\times m} \\
\Big\downarrow \text{permutation} & & \Big\downarrow \mu_m \\
 & & X \\
 & & \Big\uparrow \mu_n \\
\mathcal{P}(n) \times \Big(\mathcal{P}(m_1) \times X^{\times m_1} \cdots \times \mathcal{P}(m_n) \times X^{\times m_n}\Big) & \xrightarrow{Id \times \mu_{m_1} \times \cdots \times \mu_{m_n}} & \mathcal{P}(n) \times X^{\times n}
\end{array}
$$

Here, $m = \sum_{i=1}^{n} m_i$.

Homomorphisms of $\mathcal{P}$-algebras

Let (X,λ) and (Y,μ) be $\mathcal{P}$-algebras.

A *homomorphism* of $\mathcal{P}$-algebras from (X,λ) to (Y,μ) is a function $f : X \to Y$ such that

$$\begin{array}{ccc} \mathcal{P}(n) \times X^{\times n} & \xrightarrow{\lambda_n} & X \\ {\scriptstyle Id\times f^{\times n}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{P}(n) \times Y^{\times n} & \xrightarrow{\mu_n} & Y \end{array}$$

commutes for all n.

$\mathcal{C}$-algebras

Let (X, μ) be a $\mathcal{C}$-algebra.

For all n, there is a unique n-ary operation

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$$\widetilde{\mu}_2 : X^{\times 2} \to X$$

that must be commutative (by unicity of the binary operation) and associative (by unicity of the ternary operation).

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Conclusion: A $\mathcal{C}$-algebra is a commutative monoid.

$\mathcal{A}$-algebras

Let (X, μ) be a $\mathcal{A}$-algebra.

For all n, there is an n-ary operation

$$\mu_n(\sigma, -) : X^{\times n} \to X$$

for each permutation $\sigma \in \mathfrak{S}_n$.

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that must be associative (by equivariance).

Conclusion: An $\mathcal{A}$-algebra is a monoid.

Operads as monoids

More about symmetric sequences

Let $\mathcal{X}$ and $\mathcal{Y}$ be symmetric sequences.

Let $\mathcal{X} \circ \mathcal{Y}$ be the symmetric sequence with

$$\big(\mathcal{X} \circ \mathcal{Y}\big)(n) = \coprod_{k,\vec{n}} \mathcal{X}(k) \underset{\mathfrak{S}_k}{\times} \big(\mathcal{Y}(n_1) \times \cdots \times \mathcal{Y}(n_k)\big) \underset{\mathfrak{S}_{\vec{n}}}{\times} \mathfrak{S}_n.$$

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A *homomorphism* of symmetric sequences $\varphi : \mathcal{X} \to \mathcal{Y}$ consists of a family of equivariant functions

$$\{\varphi_n : \mathcal{X}(n) \to \mathcal{Y}(n) \mid n \geq 0\}.$$

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Proposition. An operad is a monoid with respect to the composition product of symmetric sequences.

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I.e., an operad consists of a symmetric sequence $\mathcal{P}$ together with a homomorphism of symmetric sequences $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ such that

$$\begin{array}{ccc} \mathcal{P} \circ \mathcal{P} \circ \mathcal{P} & \xrightarrow{Id_{\mathcal{P}} \circ \gamma} & \mathcal{P} \circ \mathcal{P} \\ {\scriptstyle \gamma \circ Id_{\mathcal{P}}} \downarrow & & \downarrow {\scriptstyle \gamma} \\ \mathcal{P} \circ \mathcal{P} & \xrightarrow{\gamma} & \mathcal{P} \end{array}$$

commutes.

$\mathcal{P}$-modules

A *left* $\mathcal{P}$*-module* is a symmetric sequence $\mathcal{M}$ together with a homomorphism of symmetric sequences $\lambda : \mathcal{P} \circ \mathcal{M} \to \mathcal{M}$ such that

$$\begin{array}{ccc} \mathcal{P} \circ \mathcal{P} \circ \mathcal{M} & \xrightarrow{Id \circ \lambda} & \mathcal{P} \circ \mathcal{M} \\ {\scriptstyle \gamma \circ Id} \downarrow & & \downarrow {\scriptstyle \lambda} \\ \mathcal{P} \circ \mathcal{M} & \xrightarrow{\lambda} & \mathcal{M} \end{array}$$

commutes.

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commutes.

There is an analogous dual definition of *right* $\mathcal{P}$*-modules* $(\mathcal{M}, \rho)$, leading to the definition of $\mathcal{P}$-bimodules, in which the right and left actions of $\mathcal{P}$ commute.

$\mathcal{P}$-algebras as $\mathcal{P}$-modules

Proposition. A $\mathcal{P}$-algebra naturally gives rise to a left $\mathcal{P}$-module.

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Given a set X, let $c(X)$ denote the symmetric sequence with $c(X)(n) = X$ for all n.

$\mathcal{P}$-algebras as $\mathcal{P}$-modules

Proposition. A $\mathcal{P}$-algebra naturally gives rise to a left $\mathcal{P}$-module.

Given a set X, let $c(X)$ denote the symmetric sequence with $c(X)(n) = X$ for all n. Then X admits a $\mathcal{P}$-algebra structure iff there is a homomorphism of symmetric sequences

$$\mu : \mathcal{P} \circ c(X) \to c(X)$$

such that

$$\begin{array}{ccc} \mathcal{P} \circ \mathcal{P} \circ c(X) & \xrightarrow{Id \circ \mu} & \mathcal{P} \circ c(X) \\ {\scriptstyle \gamma \circ Id}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{P} \circ c(X) & \xrightarrow{\mu} & c(X) \end{array}$$

commutes.

Homomorphisms

Let (A, μ) and (A', μ') be $\mathcal{P}$-algebras.

The existence of a homomorphism $f : A \to A'$ of $\mathcal{P}$-algebras is equivalent to the existence of a homomorphism of left $\mathcal{P}$-modules $c(A) \to c(A')$.

Generalization

In all that I have presented, we can replace sets, functions and cartesian products by:

- topological spaces, continuous maps and cartesian products;
- vector spaces, linear maps and tensor products;
- chain complexes, chain maps and graded tensor products;
- simplicial sets, simplicial maps and levelwise products.

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- topological spaces, continuous maps and cartesian products;
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- chain complexes, chain maps and graded tensor products;
- simplicial sets, simplicial maps and levelwise products.

In fact, there are notions of *monoid* and *module* in any *monoidal category*, while the notion of *operad* makes sense in any *cocomplete, closed, symmetric monoidal category*.

Bimodules and homomorphisms

Weak homomorphisms

Let $\sim$ be an equivalence relation on the set of maps $X \to Y$ for any fixed X and Y (e.g., homotopy of continuous maps, chain homotopy of chain maps).

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A *weak $\mathcal{P}$-homomorphism* from a $\mathcal{P}$-algebra (X, λ) to a $\mathcal{P}$-algebra (Y, μ) is a map $f : X \to Y$ such that

$$\begin{array}{ccc} \mathcal{P}(n) \times X^{\times n} & \xrightarrow{\lambda_n} & X \\ {\scriptstyle Id \times f^{\times n}} \downarrow & & \downarrow {\scriptstyle f} \\ \mathcal{P}(n) \times Y^{\times n} & \xrightarrow{\mu_n} & Y \end{array}$$

commutes *up to equivalence*, i.e., $f\lambda_n \sim \mu_n(Id \times f^{\times n})$, for all n.

Examples of weak homomorphisms

- Let (X, λ) and (Y, μ) be monoids of topological spaces (e.g., topological groups). A continuous map $f : X \to Y$ is a weak $\mathcal{A}$-homomorphism if $f\lambda$ is homotopic to $\mu(f \times f)$.
- Let (A, λ) and (B, μ) be monoids of chain complexes (i.e., differential graded associative algebras). A chain map $\varphi : A \to B$ is a weak $\mathcal{A}$-homomorphism if $\varphi\lambda$ is chain-homotopic to $\mu(\varphi \otimes \varphi)$.

We often go much farther, working with infinite families of homotopies.

Bimodules and weak homomorphisms

Let $(\mathcal{M}, \lambda, \rho)$ be a $\mathcal{P}$-bimodule. Let (A, μ) and (A', μ') be $\mathcal{P}$-algebras.

A $\mathcal{P}$-algebra homomorphism *up to* $\mathcal{M}$ is a homomorphism of left $\mathcal{P}$-bimodules

$$\varphi : \mathcal{M} \underset{\mathcal{P}}{\circ} c(A) \to c(A').$$

Many interesting types of weak homomorphisms can be conveniently described in terms of an appropriate bimodule.

Unraveling the definition

A homomorphism of left $\mathcal{P}$-bimodules

$$\varphi : \mathcal{M} \underset{\mathcal{P}}{\circ} c(A) \to c(A')$$

is equivalent to a “suitably coherent" family of equivariant maps

$$\{\mathcal{M}(n) \times A^{\times n} \to A' \mid n \geq 0\}.$$

Questions

- Given a certain type of weak homomorphism, can it be described in terms of an appropriate bimodule?
- Given a bimodule $\mathcal{M}$ over an operad, what is the meaning of $\mathcal{P}$-algebra homomorphisms up to $\mathcal{M}$?