THE COBAR CONSTRUCTION: A MODERN PERSPECTIVE

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Abstract. These are lecture notes from a minicourse given at Louvain-la-Neuve in May 2007. There is a high, but not perfect, correlation between the contents of these notes and the subjects covered in the minicourse.

Note that since these are lecture notes and not an article, many proofs are not included, while those that are included are only sketches, which are particularly brief when the proofs have already appeared in published articles or in articles available on the arXiv.

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Notation 0.1. Let $\mathbf{C}$ be a small category, and let $A, B \in \operatorname{Ob}\mathbf{C}$. In these notes, the set of morphisms from A to B is denoted $\mathbf{C}(A, B)$. The identity morphisms on an object A will often be denoted A as well.

1. The classical cobar construction

Notation 1.1. Let R denote a fixed commutative ring. Unless stated otherwise, $\otimes$ denotes (perhaps graded) tensor product over R.

- $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{A}$ are the categories of graded modules, of coaugmented chain coalgebras and of augmented chain algebras over R.
- T is the free augmented algebra functor on graded R-modules:
$$TV := R \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots ,$$
where multiplication is given by concatenation. Simplifying notation somewhat, an element of TV coming from the summand $V^{\otimes n}$ is denoted $v_1 \cdots v_n$, where $v_i \in V$ for all i. Note that for all $A \in \mathbf{A}$,
$$\mathbf{A}(TV, A) \cong \mathbf{M}(V, UA),$$
where $U : \mathbf{A} \to \mathbf{M}$ is the forgetful functor.
- Let C be a chain coalgebra with coaugmentation $\eta : R \to C$. The *coaugmentation coideal* of C is then
$$\overline{C} = \operatorname{coker}(\eta : R \to C).$$
- Given a graded R-module V, its *desuspension* $s^{-1}V$ is the graded R-module defined by $(s^{-1}V)_n = V_{n+1}$ for all n. The element of $s^{-1}V$ corresponding to $v \in V$ is denoted $s^{-1}v$.
- Given a coaugmented chain coalgebra C with comultiplication Δ, its *reduced comultiplication* $\overline{\Delta}$ is the composite
$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{q \otimes q} \overline{C} \otimes \overline{C},$$
where q is the quotient map.

1.1. Definition of the classical cobar construction.

Definition 1.2. The *cobar construction functor* $\Omega : \mathbf{C} \to \mathbf{A}$ is defined as follows.

- $\Omega C := \big(T(s^{-1}\overline{C}), d_\Omega\big)$ for all $C \in \operatorname{Ob}\mathbf{C}$, where d_Ω is the derivation specified by
$$d_\Omega s^{-1} = -s^{-1}d + (s^{-1} \otimes s^{-1})\overline{\Delta},$$
where d and Δ are the differential and comultiplication on C, i.e.,
$$d_\Omega(s^{-1}c) = -s^{-1}(dc) + \ (-1)^{\deg c_i} s^{-1}c_i s^{-1}c^i,$$
where $\overline{\Delta}(c) = c_i \otimes c^i$ (using Einstein notation for sums).
- $f \in \mathbf{C}(C, C')$ induces $\Omega f : \Omega C \to \Omega C'$, specified by $\Omega f(s^{-1}c) = s^{-1}f(c)$.

Remark 1.3. Ωf is indeed a differential map, since $fd = d'f$ and $(f \otimes f)\Delta = \Delta' f$.

1.2. **Twisting cochains.**

Definition 1.4. Let (C, d) be a chain coalgebra with comultiplication Δ, and let (A, d) be a chain algebra with product μ. A *twisting cochain* from (C, d) to (A, d) is a degree -1 map $t : C \to A$ of graded modules such that

$$dt + td = \mu(t \otimes t)\Delta.$$

Example 1.5. The universal example:

$$t_\Omega : C \to \Omega C : c \mapsto s^{-1}c.$$

Example 1.6. In 1961, Szczarba defined a natural twisting cochain

$$sz : C_*K \to C_*GK,$$

where

- K is a reduced simplicial set;
- GK is the the Kan loop group on K (the simplicial analogue of the based loop space functor);
- $C_*(-)$ is the normalized chains functor.

Proposition 1.7. *Let C be a chain coalgebra, and let A be a chain algebra. There is a functorial, bijective correspondance*

$$\{\textit{twisting cochains } t : C \to A\} \longleftrightarrow \mathbf{A}(\Omega C, A)$$

where

- $t : C \to A$ *gives rise to* $\theta_t : \Omega C \to A$ *determined by* $\theta_t(s^{-1}c) = t(c)$*;*
- $\theta : \Omega C \to A$ *gives rise to* $t_\theta : C \to A$*, given by the composite*

$$C \xrightarrow{t_\Omega} \Omega C \xrightarrow{\theta} A.$$

Example 1.8. The universal example: Recall $t_\Omega : C \to \Omega C : c \mapsto s^{-1}c$.

Observe that

$$\theta_{t_\Omega} = Id : \Omega C \to \Omega C,$$

i.e.,

$$t_{Id} = t_\Omega : C \to \Omega C.$$

Furthermore, t_Ω truly is universal: for all twisting cochains $t : C \to A$,

$$\begin{tikzcd} C \arrow[r, "t_\Omega"] \arrow[dr, "t"'] & \Omega C \arrow[d, "\theta"] \\ & A \end{tikzcd}$$

commutes if and only if $\theta = \theta_t$.

Example 1.9. Szczarba's twisting cochain: Szczarba proved that his twisting cochain induced a quasi-isomorphism of chain algebras

$$Sz = \theta_{sz} : \Omega C_*K \xrightarrow{\simeq} C_*GK.$$

Definition 1.10. Let $t : C \to A$ be a twisting cochain. Let N be a left C-comodule, with coaction $\lambda : N \to C \otimes N$, and let M be a right A-module, with action $\rho : M \otimes A \to M$. The *twisted extension of M by N* is

$$(M, d) \hookrightarrow (M, d) \otimes_t (N, d) = (M \otimes N, D_t) \twoheadrightarrow (N, d),$$

where

$$D_t = d \otimes N + M \otimes d + (\rho \otimes N)(M \otimes t \otimes N)(M \otimes \lambda).$$

Similarly, given a right coaction $\rho : N \to N \otimes C$ and a left action $\lambda : A \otimes M \to M$, there is a twisted extension

$$(M, d) \hookrightarrow (N, d) \otimes_t (M, d) = (N \otimes M, D_t) \twoheadrightarrow (N, d),$$

where

$$D_t = d \otimes M + N \otimes d + (N \otimes \lambda)(N \otimes t \otimes M)(\rho \otimes N).$$

Remark 1.11. The twisted tensor product $(M, d) \otimes_t (N, d)$ is a left A-module if M is an A-bimodule and is a right C-comodule if N is a C-bicomodule. In particular, $(A, d) \otimes_t (C, d)$ is a left A-module and a right C-comodule.

Example 1.12. The universal example again: $(\Omega C, d_\Omega) \otimes_{t_\Omega} (C, d)$ is the usual acyclic cobar construction. We expand on this example in the next subsection.

1.3. **The cobar construction and the cotensor product.**

Definition 1.13. Let $C \in \operatorname{Ob} \mathbf{C}$, and let (N, ρ) and (N', λ) be a right and a left differential C-comodule, respectively. The *cotensor product* of N and N' over C, denoted $N \underset{C}{\square} N'$ is the equalizer of

$$N \otimes N' \underset{N \otimes \lambda}{\overset{\rho \otimes N'}{\rightrightarrows}} N \otimes C \otimes N',$$

i.e.,

$$N \underset{C}{\square} N' = \ker(\rho \otimes N' - N \otimes \lambda).$$

Exercise 1.14. If the underlying graded R-module of (N, ρ) is a cofree right C-comodule, i.e., $N = X \otimes C$ and $\rho = X \otimes \Delta$, and (N', λ) is any left C-comodule, then $N \underset{C}{\square} N' \cong X \otimes N'$, as graded R-modules.

Exercise 1.15. Consider R endowed with the trivial left C-comodule structure. Then for any right C-comodule (N, ρ),

$$N \underset{C}{\square} I = \{x \in N \mid \rho(x) = x \otimes 1\},$$

i.e., $N \underset{C}{\square} I$ consists of the "fixed points" of the coaction ρ.

Exercise 1.16. The cotensor product is not homotopy invariant. For example, there are quasi-isomorphisms

$$R \xrightarrow{\simeq} \Omega C \otimes_{t_\Omega} C \quad \text{and} \quad R \xrightarrow{\simeq} C \otimes_{t_\Omega} \Omega C$$

of right and left C-comodules, respectively, but

$$R \underset{C}{\square} R \cong R,$$

while

$$(\Omega C \otimes_{t_\Omega} C) \underset{C}{\square} (C \otimes_{t_\Omega} \Omega C) \cong \Omega C \otimes_{t_\Omega} C \otimes_{t_\Omega} \Omega C \simeq \Omega C,$$

which is usually not acyclic!

Definition 1.17. Let $C \in \operatorname{Ob} \mathbf{C}$ with comultiplication Δ, and let (N, ρ) and (N', λ) be a right and a left differential C-comodule, respectively. The *homotopy cotensor product* of N and N' over C, denoted $N \underset{C}{\widehat{\square}} N'$ is the equalizer of

$$(N \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} C) \otimes (C \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} N') \underset{Id \otimes \hat{\lambda}}{\overset{\hat{\rho} \otimes Id}{\rightrightarrows}} (N \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} C) \otimes C \otimes (C \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} N'),$$

where $\hat{\rho} = N \otimes \Omega C \otimes \Delta$ and $\hat{\lambda} = \Delta \otimes \Omega C \otimes N'$.

Exercise 1.18. There is a natural quasi-isomorphism of chain complexes

$$N \underset{C}{\widehat{\square}} N' \xrightarrow{\simeq} N \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} N'.$$

Notation 1.19. $\operatorname{Cotor}_*^C(N, N') := H_*(N \underset{C}{\widehat{\square}} N')$.

Proposition 1.20. *The homotopy cotensor product is homotopy invariant, up to homotopy, i.e., given quasi-isomorphisms $f : N \xrightarrow{\simeq} P$ and $f' : N' \xrightarrow{\simeq} P'$ of right and left C-comodules with R-free underlying graded modules, respectively, there is an induced quasi-isomorphism of chain complexes*

$$f \underset{C}{\widehat{\square}} f' : N \underset{C}{\widehat{\square}} N' \xrightarrow{\simeq} P \underset{C}{\widehat{\square}} P'.$$

In particular, $\operatorname{Cotor}_*^C(N, N') \cong \operatorname{Cotor}_*^C(P, P')$.

Proof. The quasi-isomorphisms f and f' induce a quasi-isomorphism

$$f \otimes \Omega C \otimes f' : N \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} N' \xrightarrow{\simeq} P \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} P'.$$

(Spectral sequence argument.) □

1.4. **The Milgram equivalence.**

Definition 1.21. Let $t : C \to A$ and $t' : C' \to A'$ be twisting cochains. Consider

$$C \xrightarrow{\varepsilon} R \xrightarrow{\eta} A \quad \text{and} \quad C' \xrightarrow{\varepsilon'} R \xrightarrow{\eta'} A'.$$

The *convolution* of t and t' is the twisting cochain

$$t * t' := t \otimes \eta' \varepsilon' + \eta \varepsilon \otimes t' : C \otimes C' \to A \otimes A'.$$

Exercise 1.22. $t * t'$ really is a twisting cochain.

Notation 1.23. If $\theta : \Omega C \to A$ and $\theta' : \Omega C' \to A'$ are the chain algebra maps induced by twisting cochains t and t', then

$$\theta * \theta' : \Omega(C \otimes C') \to A \otimes A'$$

is the chain algebra map induced by $t * t'$.

Definition 1.24. Let $C, C' \in \operatorname{Ob} \mathbf{C}$. Consider the convolution

$$t_\Omega * t_\Omega : C \otimes C' \to \Omega C \otimes \Omega C'.$$

The associated chain algebra map

$$q := Id * Id : \Omega(C \otimes C') \to \Omega C \otimes \Omega C'$$

is the *Milgram map*, which is the (C, C')-component of a natural transformation

$$q : \Omega(- \otimes -) \to \Omega(-) \otimes \Omega(-)$$

of functors from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{A}$.

Milgram's classical result...

Theorem 1.25 (Milgram)**.** *If C and C' are connected ($C_0 = R = C'_0$) chain coalgebras, then the Milgram map $q : \Omega(C \otimes C') \to \Omega C \otimes \Omega C'$ is a quasi-isomorphism.*

An even stronger and more general result can be be proved in terms of strong deformation retracts, important because of their role in homological perturbation theory.

Definition 1.26. Suppose that $\nabla : (X, \partial) \to (Y, d)$ and $f : (Y, d) \to (X, \partial)$ are morphisms of chain complexes. If $f\nabla = 1_X$ and there exists a chain homotopy $\varphi : (Y, d) \to (Y, d)$ such that

(1) $d\varphi + \varphi d = \nabla f - 1_Y$,
(2) $\varphi\nabla = 0$,
(3) $f\varphi = 0$, and
(4) $\varphi^2 = 0$,

then $(X, d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y, d) \circlearrowleft \varphi$ is a *strong deformation retract (SDR) of chain complexes.*

It is easy to show that given a chain homotopy φ' satisfying condition (1), there exists a chain homotopy φ satisfying all four conditions:

$$\varphi = (\nabla f - Y)\varphi'(\nabla f - Y)d(\nabla f - Y)\varphi'(\nabla f - Y).$$

Theorem 1.27 (H.-Parent-Scott)**.** *If C and C' are coaugmented, then there is a strong deformation retract of chain complexes*

$$\Omega C \otimes \Omega C' \underset{q}{\overset{\sigma}{\rightleftarrows}} \Omega(C \otimes C') \circlearrowleft \varphi.$$

In particular, if C and C' are coaugmented, then q is a quasi-isomorphism.

1.5. **The category DCSH.**

Question 1.28. As seen above, $f \in \mathbf{C}(C, C')$ always induces $\Omega f \in \mathbf{A}(\Omega C, \Omega C')$. There are, however, many morphisms $\varphi \in \mathbf{A}(\Omega C, \Omega C')$ such that $\varphi \neq \Omega f$ for all $f \in \mathbf{C}(C, C')$. What is the significance of these maps?

Definition 1.29. Let **DCSH** denote the category with

- $\mathrm{Ob}\,\mathbf{DCSH} = \mathrm{Ob}\,\mathbf{C}$, and
- $\mathbf{DCSH}(C, C') := \mathbf{A}(\Omega C, \Omega C')$.

Morphisms in **DCSH** are *strongly homotopy-comultiplicative maps.*

(First introduced by Gugenheim and Munkholm, when they were studying extended naturality of Cotor.)

Remark 1.30. Unraveling the definition of the cobar construction, we see that

$$\varphi \in \mathbf{DCSH}(C, C') \Longleftrightarrow \mathfrak{F}(\varphi) := \{\varphi_k : C \to (C')^{\otimes k}\}_{k \geq 1},$$

where φ_k is homogeneous of degree $k-1$ and (up to signs)

$$d_{(C')^{\otimes k}}\varphi_k \pm \varphi_k d_C = \sum_{i+j=k} \pm(\varphi_i \otimes \varphi_j)\Delta_C + \sum_{i+j=k-2} \pm\big((C')^{\otimes i} \otimes \Delta_{C'} \otimes (C')^{\otimes j}\big)\varphi_{k-1}.$$

In particular, $\varphi_1 : C \to C'$ is a chain map, which is a coalgebra map up to chain homotopy, given by φ_2.

Notation 1.31. Given $\varphi \in \mathbf{DCSH}(C, C')$ with corresponding family $\mathfrak{F}(\varphi)$, the chain map $\varphi_1 : C \to C'$ is called a *DCSH-map*.

The essential topological reason for the importance of **DCSH**...

Theorem 1.32 (Gugenheim-Munkholm)**.** *Let K be a reduced simplicial set. The natural comultiplication $\Delta_K : C_*K \to C_*K \otimes C_*K$ is naturally a DCSH-map, i.e., there exists*

$$\varphi_K \in \mathbf{A}\Big(\Omega C_*K, \Omega\big(C_*K \otimes C_*K\big)\Big),$$

natural in K, such that $(\varphi_K)_1 = \Delta_K$.

Theorem 1.33. *The composite chain algebra map*

$$\Omega C_*K \xrightarrow{\varphi_K} \Omega(C_*K \otimes C_*K) \xrightarrow{q} \Omega C_*K \otimes \Omega C_*K,$$

denoted ψ_K, endows ΩC_*K with a natural chain Hopf algebra structure.

Proof. Some fairly tough calculations to show that ψ_K is coassociative. □

Terminology 1.34. We call ψ_K the *Alexander-Whitney diagonal.*

Remark 1.35. It turns out that ψ_K is the same as a comultiplication map defined purely combinatorially by Baues.

Question 1.36. Topological relevance of the Alexander-Whitney comultiplication? To be answered in Lecture 3.

Theorem 1.32 follows from a more general result.

Definition 1.37. An SDR $(X, d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y, d) \circlearrowleft \varphi$ is called *Eilenberg-Zilber (E-Z) data* if (Y, d, Δ_Y) and (X, d, Δ_X) are chain coalgebras and ∇ is a morphism of coalgebras.

Theorem 1.38 (Gugenheim-Munkholm)**.** *Let $(X, d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y, d) \circlearrowleft \varphi$ be E-Z data such that Y is simply connected and X is connected. Let $F_1 = f$. Given F_i for all $i < k$, let*

$$F_k = -\sum_{i+j=k} (F_i \otimes F_j)\Delta_Y \varphi.$$

Similarly, let $\Phi_1 = \varphi$, and, given Φ_i for all $i < k$, let

$$\Phi_k = \big(\Phi_{k-1} \otimes 1_Y + \sum_{i+j=k} \nabla^{\otimes i} F_i \otimes \Phi_j\big)\Delta_Y \varphi.$$

Then

$$\Omega(X, d) \underset{\widetilde{\Omega} f}{\overset{\Omega\nabla}{\rightleftarrows}} \Omega(Y, d) \circlearrowleft \widetilde{\Omega}\varphi$$

is an SDR of chain algebras, where $\widetilde{\Omega} f = \sum_{k \geq 1} (s^{-1})^{\otimes k} F_k s$ and $\widetilde{\Omega}\varphi = \sum_{k \geq 1} (s^{-1})^{\otimes k} \Phi_k s$.

To prove Theorem 1.32, we apply Theorem 1.38 to the *Eilenberg-Zilber SDR*, described in the next example.

Example 1.39. Let K and L be two simplicial sets. Define morphisms on their normalized chain complexes

$$\nabla_{K,L} : C(K) \otimes C(L) \to C(K \times L) \quad \text{and} \quad f_{K,L} : C(K \times L) \to C(K) \otimes C(L)$$

by

$$\nabla_{K,L}(x \otimes y) = \sum_{(\mu,\nu)\in \mathcal{S}_{p,q}} (-1)^{\mathrm{sgn}(\mu)}(s_{\nu_q}...s_{\nu_1}x, s_{\mu_p}...s_{\mu_1}y)$$

where $\mathcal{S}_{p,q}$ denotes the set of (p,q)-shuffles, $\mathrm{sgn}(\mu)$ is the signature of μ and $x \in K_p$, $y \in L_q$, and

$$f_{K,L}((x,y)) = \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n x \otimes \partial_0^i y$$

where $(x,y) \in (K \times L)_n$. We call $\nabla_{K,L}$ the *shuffle (or Eilenberg-Zilber) map* and $f_{K,L}$ the *Alexander-Whitney map.* There is a chain homotopy, $\varphi_{K,L}$, so that

$$C(K) \otimes C(L) \underset{f_{K,L}}{\overset{\nabla_{K,L}}{\rightleftarrows}} C(K \times L) \circlearrowleft \varphi_{K,L}$$

is an SDR of chain complexes. Furthermore $\nabla_{K,L}$ is a map of coalgebras, with respect to the usual coproducts, which are defined in terms of the natural equivalence $f_{K,L}$.

2. An operadic approach to the cobar construction

2.1. Operads of chain complexes and their co-rings.

Notation 2.1. **Ch** is the category of chain complexes concentrated in non-negative degrees over a commutative ring R, endowed with its usual (i.e., graded) tensor product.

Definition 2.2. $\mathbf{Ch}^\Sigma$ is the *category of symmetric sequences* in **Ch**. An object $\mathcal{X}$ of $\mathbf{Ch}^\Sigma$ is a family $\{\mathcal{X}(n) \in \mathbf{Ch} \mid n \geq 0\}$ of objects in **Ch** such that $\mathcal{X}(n)$ admits a right action of the symmetric group Σ_n, for all n. The object $\mathcal{X}(n)$ is called the n^{th} *level* of the symmetric sequence $\mathcal{X}$.

For all $\mathcal{X}, \mathcal{Y} \in \mathbf{Ch}^\Sigma$, a *morphism of symmetric sequences* $\varphi : \mathcal{X} \to \mathcal{Y}$ consists of a family

$$\{\varphi_n \in \mathbf{C}\big(\mathcal{X}(n), \mathcal{Y}(n)\big) \mid \varphi_n \quad \text{is } \Sigma_n\text{-equivariant}, n \geq 0\}.$$

More formally, $\mathbf{Ch}^\Sigma$ is the category of contravariant functors from the symmetric groupoid $\mathbf{\Sigma}$ to $\mathbf{C}$, where $\mathrm{Ob}\,\mathbf{\Sigma} = \mathbb{N}$, the set of natural numbers, and $\mathbf{\Sigma}(m,n)$ is empty if $m \neq n$, while $\mathbf{\Sigma}(n,n) = \Sigma_n$.

Definition 2.3. The *tensor embedding* of **Ch** into $\mathbf{Ch}^\Sigma$

$$\mathcal{T} : \mathbf{Ch} \to \mathbf{Ch}^\Sigma \tag{2.1}$$

is defined by $\mathcal{T}(A)(n) = A^{\otimes n}$ for all n. The right action of Σ_n on $\mathcal{T}(A)(n) = A^{\otimes n}$ is given by permutation of the factors, using iterates of the natural symmetry isomorphism

$$\tau : A \otimes A \xrightarrow{\cong} A \otimes A : a \otimes a' \mapsto (-1)^{\deg a \deg a'} a' \otimes a$$

in **Ch**.

Definition 2.4. The *level tensor product* of two symmetric sequences $\mathcal{X}$ and $\mathcal{Y}$ is the symmetric sequence given by

$$(\mathcal{X} \otimes \mathcal{Y})(n) = \mathcal{X}(n) \otimes \mathcal{Y}(n) \quad (n \geq 0),$$

endowed with the diagonal action of Σ_n.

Proposition 2.5. *Let $\mathcal{C} = \{\mathcal{C}(n)\}_{n\geq 0}$ be the symmetric sequence with $\mathcal{C}(n) = R$ and trivial Σ_n-action, for all $n \geq 0$. Then $(\mathbf{Ch}^\Sigma, \otimes, \mathcal{C})$ is a closed symmetric monoidal category, called the* level monoidal structure *on $\mathbf{Ch}^\Sigma$.*

Terminology 2.6. A (co)monoid in $\mathbf{Ch}^\Sigma$ with respect to the level monoidal structure, i.e., a symmetric sequence $\mathcal{X}$ endowed with a coassociative comultiplication $\mathcal{X} \to \mathcal{X} \otimes \mathcal{X}$ that is counital with respect to a morphism $\mathcal{X} \to \mathcal{C}$, is called a *level (co)monoid*.

Remark 2.7. The functor $\mathcal{T}$ is strong monoidal, i.e., for all $C, C \in \operatorname{Ob} \mathbf{C}$, there is a natural isomorphism $\mathcal{T}(C \otimes C') \cong \mathcal{T}(C) \otimes \mathcal{T}(C')$, again given in each level by iterated application of the natural symmetry isomorphism in $\mathbf{C}$.

Definition 2.8. The *composition tensor product* of two symmetric sequences $\mathcal{X}$ and $\mathcal{Y}$ is the symmetric sequence $\mathcal{X} \diamond \mathcal{Y}$ given by

$$(\mathcal{X} \diamond \mathcal{Y})(n) = \coprod_{\substack{k\geq 1\\ \vec{n}\in I_{k,n}}} \mathcal{X}(k) \underset{\Sigma_k}{\otimes} \big(\mathcal{Y}(n_1) \otimes \cdots \otimes \mathcal{Y}(n_k)\big) \underset{\Sigma_{\vec{n}}}{\otimes} R[\Sigma_n],$$

where $I_{k,n} = \{\vec{\imath} = (n_1, ..., n_k) \in \mathbb{N}^k \mid \sum_j n_j = n\}$ and $\Sigma_{\vec{n}} = \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$, seen as a subgroup of Σ_n. The left action of Σ_k on $\coprod_{\vec{n}\in I_{k,n}} \mathcal{Y}(n_1) \otimes \cdots \otimes \mathcal{Y}(n_k)$ is given by permutation of the factors, using the natural symmetry isomorphism $A \otimes B \cong B \otimes A$ in $\mathbf{Ch}$.

Proposition 2.9. *Let $\mathcal{J}$ denote the symmetric sequence with $\mathcal{J}(1) = R$ and $\mathcal{J}(n) = 0$ otherwise, with trivial Σ_n-action. Then $(\mathbf{Ch}^\Sigma, \diamond, \mathcal{J})$ is a right-closed monoidal category, called the* composition monoidal structure *on $\mathbf{Ch}^\Sigma$.*

Remark 2.10. For any objects $\mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}'$ in $\mathbf{Ch}^\Sigma$, there is an obvious, natural *intertwining* map

$$\mathfrak{i} : (\mathcal{X} \otimes \mathcal{X}') \diamond (\mathcal{Y} \otimes \mathcal{Y}') \longrightarrow (\mathcal{X} \diamond \mathcal{Y}) \otimes (\mathcal{X}' \diamond \mathcal{Y}') .$$

Definition 2.11. An *operad* of chain complexes is a monoid in $\mathbf{Ch}^\Sigma$ with respect to the composition product, i.e., a triple $(\mathcal{P}, \gamma, \eta)$, where $\gamma : \mathcal{P} \diamond \mathcal{P} \to \mathcal{P}$ and $\eta : \mathcal{J} \to \mathcal{P}$ are morphisms in $\mathbf{C}^\Sigma$, and γ is appropriately associative and unital with respect to η. A morphism of operads is a monoid morphism in the category of symmetric sequences.

Example 2.12. The most important example of an operad for these lectures is the *associative operad* $\mathcal{A}$, given by $\mathcal{A}(n) = R[\Sigma_n]$ for all n, endowed with the obvious multiplication map, induced by permutation of blocks.

Definition 2.13. A $\mathcal{P}$-*coalgebra* is an object A of $\mathbf{Ch}$ along with a sequence of structure morphisms

$$\theta_n : A \otimes \mathcal{P}(n) \to A^{\otimes n}, \quad n \geq 0$$

that are appropriately associative, equivariant, and unital.

A morphism of $\mathcal{P}$-coalgebras is a morphism in **Ch** that commutes with the coalgebra structure maps. The category of $\mathcal{P}$-coalgebras and their morphisms is denoted $\mathcal{P}$-**Coalg**.

Remark 2.14. $\mathcal{A}$-**Coalg** is the category of coassociative chain coalgebras. The category **C** of coaugmented, coassociative coalgebras of Lecture 1 is then exactly the undercategory (or coslice category) $R \downarrow \mathcal{A}$-**Coalg**.

Definition 2.15. Let $\mathcal{P}$ be an operad of chain complexes. A *right (resp. left) $\mathcal{P}$-module* is a symmetric sequence $\mathcal{M}$ endowed with an action $\rho : \mathcal{M} \diamond \mathcal{P} \to \mathcal{M}$ (resp. $\lambda : \mathcal{P} \diamond \mathcal{M} \to \mathcal{M}$). If $\mathcal{M}$ is endowed with right and left $\mathcal{P}$-actions that commute with each other, then it is a $\mathcal{P}$-bimodule.

Notation 2.16. $\mathbf{Mod}_{\mathcal{P}}$ is the category of right $\mathcal{P}$-modules and of morphisms of symmetric sequences between them, which respect the $\mathcal{P}$-action.

Proposition 2.17. *The tensor embedding restricts to*

$$\mathcal{T} : \mathcal{P}\text{-}\mathbf{Coalg} \longrightarrow \mathbf{Mod}_{\mathcal{P}}$$

from the category of $\mathcal{P}$-coalgebras to the category of right $\mathcal{P}$-modules, i.e, $\mathcal{P}$-coalgebra structure on an object C in **Ch** *induces a right $\mathcal{P}$-action map on $\mathcal{T}(C)$ in* $\mathbf{Ch}^{\Sigma}$.

Remark 2.18. Given a right $\mathcal{P}$-module $\mathcal{M}$ and a left $\mathcal{P}$-module $\mathcal{N}$, define $\mathcal{M} \underset{\mathcal{P}}{\diamond} \mathcal{N}$ to be the coequalizer (calculated in $\mathbf{Ch}^{\Sigma}$) of

$$\mathcal{M} \diamond \mathcal{P} \diamond \mathcal{N} \underset{\mathcal{M}\diamond\lambda_{\mathcal{N}}}{\overset{\rho_{\mathcal{M}}\diamond\mathcal{N}}{\rightrightarrows}} \mathcal{M} \diamond \mathcal{N}.$$

Since $- \diamond \mathcal{X}$ admits a right adjoint for all $\mathcal{X}$, it commutes with all colimits. As a consequence, if $\mathcal{N}$ is actually a $\mathcal{P}$-bimodule, then $\mathcal{M} \underset{\mathcal{P}}{\diamond} \mathcal{N}$ inherits a right $\mathcal{P}$-action from the right $\mathcal{P}$-action on $\mathcal{N}$.

Definition 2.19. A *$\mathcal{P}$-co-ring* consists of an $\mathcal{P}$-bimodule $\mathcal{M}$ together with morphisms of $\mathcal{P}$-bimodules

$$\psi : \mathcal{M} \to \mathcal{M} \underset{\mathcal{P}}{\diamond} \mathcal{M} \quad \text{and} \quad \varepsilon : \mathcal{M} \to \mathcal{P}$$

such that ψ is coassociative and counital with respect to ε.

Definition 2.20. Let $\mathcal{P}$ be an operad of chain complexes, and let $\mathcal{M}$ be a $\mathcal{P}$-co-ring, with comultiplication ψ and counit ε. Let $(\mathcal{P}, \mathcal{M})$-**Coalg** be category with

- $\mathrm{Ob}(\mathcal{P}, \mathcal{M})$-**Coalg** $= \mathrm{Ob}\,\mathcal{P}$-**Coalg**, and
- $$(\mathcal{P}, \mathcal{M})\text{-}\mathbf{Coalg}(C, C') := \mathbf{Mod}_{\mathcal{P}}\big(\mathcal{T}(C) \underset{\mathcal{P}}{\diamond} \mathcal{M}, \mathcal{T}(C')\big), \tag{2.2}$$

  for all $C, C' \in \mathrm{Ob}(\mathcal{P}, \mathcal{M})$-**Coalg**.

Let $\varphi : \mathcal{T}(C) \underset{\mathcal{P}}{\diamond} \mathcal{M} \to \mathcal{T}(C')$ and $\varphi' : \mathcal{T}(C') \underset{\mathcal{P}}{\diamond} \mathcal{M} \to \mathcal{T}(C'')$ be morphisms of strict right $\mathcal{P}$-modules, seen as morphisms in $(\mathcal{P}, \mathcal{M})$-**Coalg** from C to C' and from C' to C'', respectively. Their composite in $(\mathcal{P}, \mathcal{M})$-**Coalg** is defined to be equal to the following composite in $\mathbf{Mod}_{\mathcal{P}}$.

$$\mathcal{T}(C) \underset{\mathcal{P}}{\diamond} \mathcal{M} \xrightarrow{\mathcal{T}(C) \underset{\mathcal{P}}{\diamond} \psi} \mathcal{T}(C) \underset{\mathcal{P}}{\diamond} \mathcal{M} \underset{\mathcal{P}}{\diamond} \mathcal{M} \xrightarrow{\varphi \underset{\mathcal{P}}{\diamond} \mathcal{M}} \mathcal{T}(C') \underset{\mathcal{P}}{\diamond} \mathcal{M} \xrightarrow{\varphi'} \mathcal{T}(C')$$

The identity morphism in $(\mathcal{P},\mathcal{M})$-**Coalg** on an object C is the following morphism in $\mathbf{Mod}_{\mathcal{P}}$.

$$\mathcal{T}(C)\underset{\mathcal{P}}{\diamond}\mathcal{M}\xrightarrow{\mathcal{T}(C)\underset{\mathcal{P}}{\diamond}\varepsilon}\mathcal{T}(C)\underset{\mathcal{P}}{\diamond}\mathcal{P}\cong\mathcal{T}(C)$$

Associativity and unitality of composition follow from coassociativity and counitality of ψ.

Remark 2.21. There is a faithful functor (natural in $\mathcal{M}$)

$$\mathfrak{I}_{\mathcal{M}}:\mathcal{P}\text{-}\mathbf{Coalg}\to(\mathcal{P},\mathcal{M})\text{-}\mathbf{Coalg},\tag{2.3}$$

which is the identity on objects and which sends a morphism $f:C\to C'$ of $\mathcal{P}$-coalgebras to

$$\mathfrak{I}_{\mathcal{M}}(f)=\mathcal{T}(f)\underset{\mathcal{P}}{\diamond}\varepsilon:\mathcal{T}(C)\underset{\mathcal{P}}{\diamond}\mathcal{M}\to\mathcal{T}(C')\underset{\mathcal{P}}{\diamond}\mathcal{P}\cong\mathcal{T}(C').$$

Definition 2.22. Let $\mathcal{P}$ be an operad of chain complexes, and let $\mathcal{M}$ be a $\mathcal{P}$-co-ring, endowed with a strict morphism of left $\mathcal{P}$-modules $\eta:\mathcal{P}\to\mathcal{M}$. Let $C,C'\in\mathcal{P}$-**Coalg**. A morphism $f\in\mathbf{Ch}(C,C')$ is a *morphism of* $\mathcal{P}$*-coalgebras up to* $\mathcal{M}$*-parametrization* if there is a strict morphism of right $\mathcal{P}$-modules

$$\varphi:\mathcal{T}(C)\underset{\mathcal{P}}{\diamond}\mathcal{M}\to\mathcal{T}(C')$$

such that the following diagram in $\mathbf{Ch}^{\Sigma}$ commutes.

$$\begin{tikzcd}
\mathcal{T}(C)\underset{\mathcal{P}}{\diamond}\mathcal{P}\cong\mathcal{T}(C) \arrow[d, "Id_{\mathcal{T}(C)}\underset{\mathcal{P}}{\diamond}\eta"'] \arrow[dr, "\mathcal{T}(f)"] & \\
\mathcal{T}(C)\underset{\mathcal{P}}{\diamond}\mathcal{M} \arrow[r, "\varphi"'] & \mathcal{T}(C')
\end{tikzcd}$$

.

Slogan 2.23. Co-rings over operads are, in a strong sense, *relative operads*. They parametrize higher, “up to homotopy” structure on morphisms of $\mathcal{P}$-coalgebras and govern relations among the higher homotopies and the n-ary cooperations on the source and target.

2.2. **An operadic description of DCSH.** Let $\mathcal{A}$ denote the associative operad in the category of chain complexes.

Definition 2.24. The *Alexander-Whitney co-ring* is an $\mathcal{A}$-co-ring $\mathcal{F}$, with compatible level comonoidal structure, which is defined as follows.

- As symmetric sequences of graded modules,
$$\mathcal{F}=\mathcal{A}\diamond\mathcal{S}\diamond\mathcal{A},$$
where, for all $n\geq 1$, $\mathcal{S}(n)=R[\Sigma_n]\cdot z_{n-1}$, the free $R[\Sigma_n]$-module on a generator of degree $n-1$, and $\mathcal{S}(0)=0$.
- Let 1 denote the generator of $\mathcal{A}(1)=R[\Sigma_1]$. The differential $\partial_{\mathcal{F}}$ on $\mathcal{F}$ is specified by
$$\begin{aligned}\partial_{\mathcal{F}}(1\otimes z_n\otimes 1^{\otimes n+1})=&\sum_{0\leq i\leq n-1}\delta\otimes(z_i\otimes z_{n-i-1})\otimes 1^{\otimes n+1}\\&+\sum_{0\leq i\leq n-1}1\otimes z_{n-1}\otimes(1^{\otimes i}\otimes\delta\otimes 1^{\otimes n-i-1}),\end{aligned}$$

where $\delta \in \mathcal{A}(2) = R[\Sigma_2]$ is a generator.

- The composition comultiplication

$$\psi_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \underset{\mathcal{A}}{\diamond} \mathcal{F} \cong \mathcal{A} \diamond \mathcal{S} \diamond \mathcal{A} \diamond \mathcal{S} \diamond \mathcal{A}$$

is specified by

$$\psi_{\mathcal{F}}(1 \otimes z_n \otimes 1^{\otimes n+1}) = \sum_{\substack{1 \leq k \leq n+1 \\ \vec{\imath} \in I_{k,n+1}}} 1 \otimes z_{k-1} \otimes 1^{\otimes k} \otimes (z_{i_1-1} \otimes \cdots \otimes z_{i_k-1}) \otimes 1^{\otimes n+1}$$

for all $n \geq 0$, where $I_{k,n} = \{\vec{\imath} = (i_1, ..., i_k) \mid \sum_j i_j = n\}$.

- The level comultiplication

$$\Delta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{F},$$

is a morphism of $\mathcal{A}$-co-rings specified by

$$\Delta_{\mathcal{F}}(1 \otimes z_n \otimes 1^{\otimes n+1}) = \tag{2.4}$$

$$\sum_{\substack{1 \leq k \leq n+1 \\ \vec{\imath} \in I_{k,n+1}}} \big(1 \otimes z_{k-1} \otimes (\delta^{(i_1)} \otimes \cdots \otimes \delta^{(i_k)})\big) \otimes \big(\delta^{(k)} \otimes (z_{i_1-1} \otimes \cdots \otimes z_{i_k-1}) \otimes 1^{\otimes n+1}\big).$$

Here, $\delta^{(i)} \in \mathcal{A}(i)$ denotes the appropriate iterated composition product of $\delta^{(2)} = \delta$.

Remark 2.25. The obvious augmentation $\varepsilon_{\mathcal{F}} : \mathcal{F} \to \mathcal{A}$ is a levelwise quasi-isomorphism of $\mathcal{A}$-bimodules.

Remark 2.26. $\mathcal{F}$ admits an obvious, increasing differential filtration, which both the composition comultiplication and the level comultiplication respect, i.e.,

> $\mathcal{F}$ is a filtered co-ring with compatible level comultiplication.

Recall that **A** denotes the category of augmented, associative chain algebras.

Theorem 2.27. *There is a full and faithful functor, called the* induction functor*,*

$$\mathrm{Ind} : (\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg} \to \mathbf{A}$$

defined on objects by $\mathrm{Ind}(C) = \Omega C$ *for all* $C \in \mathrm{Ob}(\mathcal{A}, \mathcal{F})$-**Coalg** *and on morphisms by*

$$\mathrm{Ind}(\theta) : \Omega C \to \Omega C' : s^{-1}c \mapsto \sum_{k \geq 1} (s^{-1})^{\otimes k} \theta(c \otimes z_{k-1})$$

for all $\theta \in (\mathcal{A}, \mathcal{F})$-**Coalg**$(C, C')$*.*

Proof. Straightforward calculations using the definition of $\mathcal{F}$. □

Corollary 2.28. *There is an isomorphism of categories*

$$(\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg} \xrightarrow{\cong} \mathbf{DCSH}$$

defined to be the identity on objects and to be Ind *on morphisms.*

Remark 2.29. Thanks to this operadic description of **DCSH**, we see that

> a DCSH map is exactly a morphism of coassociative coalgebras *up to $\mathcal{F}$-parametrization.*

Definition 2.30. Let $\mathbf{D}$ be a category, and let $\mathfrak{M}$ be a set of objects in $\mathbf{D}$. A functor $X : \mathbf{D} \to \mathbf{Ch}$ is *free* with respect to $\mathfrak{M}$ if there is a set $\{e_M \in X(M) \mid M \in \mathfrak{M}\}$ such that $\{X(f)(e_M) \mid f \in \mathbf{D}(M, D), M \in \mathfrak{M}\}$ is an R-basis of $X(D)$ for all objects D in $\mathbf{D}$.
If $X : \mathbf{D} \to \mathbf{E}$, where $\mathbf{E}$ admits a forgetful functor $U : \mathbf{E} \to \mathbf{Ch}$, then X is *free* with respect to $\mathfrak{M}$ if $UX : \mathbf{D} \to \mathbf{M}$ is free.

Recall that $\mathbf{C}$ denotes the category of coaugmented, coassociative chain coalgebras.

The importance of the operadic description of **DCSH** is due in large part to the existence theorems it enables us to prove. For example...

Theorem 2.31. *Let $X, Y : \mathbf{D} \to \mathbf{C}$ be functors, where $\mathbf{D}$ is a category admitting a set of models $\mathfrak{M}$ with respect to which X is free and Y is acyclic. Let $\theta : UX \to UY$ be any natural transformation of functors into $\mathbf{Ch}$, where $U : \mathbf{C} \to \mathbf{Ch}$ denotes the forgetful functor. Then there exists a natural transformation $\widehat{\theta} : \Omega X \to \Omega Y$ extending the desuspension of θ, i.e., for all objects D in $\mathbf{D}$,*

$$\widehat{\theta}_D = s^{-1}\theta_D s + \textit{higher-order terms.}$$

In other words,

> *any natural chain map $\theta_D : UX(D) \to UY(D)$ is naturally a DCSH-map.*

Proof. Argument by induction on degrees of the e_M and on filtration degree in $\mathcal{F}$, using the operadic characterization of **DCSH**. □

2.3. **Monoidal structure of DCSH.** What we need to describe DCSH structure of maps between chain Hopf algebras... The level comonoidal structure $\Delta : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ is the key!

Recall (Remark 2.10) the natural intertwining map

$$\mathfrak{i} : (\mathcal{X} \otimes \mathcal{X}') \diamond (\mathcal{Y} \otimes \mathcal{Y}') \longrightarrow (\mathcal{X} \diamond \mathcal{Y}) \otimes (\mathcal{X}' \diamond \mathcal{Y}') .$$

Definition 2.32. Let $\wedge : (\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg} \times (\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg} \to (\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg}$ denote the bifunctor defined as follows.

- $C \wedge C'$ is the usual tensor product of chain coalgebras, for all $C, C' \in \mathrm{Ob}(\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg}$.
- Given morphisms of right $\mathcal{A}$-modules $\theta : \mathcal{T}(B) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(B')$ and $\varphi : \mathcal{T}(C) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(C')$ (i.e., morphisms in $(\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg}$ from B to B' and from C to

$C')$, $\theta \wedge \varphi$ denotes the following composite.

$$\begin{tikzcd}
\mathcal{T}(B \wedge C) \underset{\mathcal{A}}{\diamond} \mathcal{F} \arrow[r, "\cong"] \arrow[rrdddd, "\theta\wedge\varphi"'] & \big(\mathcal{T}(B) \otimes \mathcal{T}(C)\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} \arrow[r, "Id \underset{\mathcal{A}}{\diamond} \Delta_{\mathcal{F}}"] & \big(\mathcal{T}(b) \otimes \mathcal{T}(C)\big) \underset{\mathcal{A}}{\diamond} (\mathcal{F} \otimes \mathcal{F}) \arrow[d, "\mathfrak{i}"] \\
& & \big(\mathcal{T}(B) \underset{\mathcal{A}}{\diamond} \mathcal{F}\big) \otimes \big(\mathcal{T}(C) \underset{\mathcal{A}}{\diamond} \mathcal{F}\big) \arrow[d, "\theta\otimes\varphi"] \\
& & \mathcal{T}(B') \otimes \mathcal{T}(C') \arrow[d, "\cong"] \\
& & \mathcal{T}(B' \wedge C')
\end{tikzcd}$$

Proposition 2.33. *The bifunctor $\wedge$ endows $(\mathcal{A}, \mathcal{F})$-**Coalg** with the structure of a symmetric monoidal category, extending the usual symmetric monoidal structure on $\mathcal{A}$-**Coalg**.*

Remark 2.34. It is rather messy to define this monoidal structure without recourse to $\mathcal{F}$.

Definition 2.35. A *pseudo Hopf algebra* is a monoid (H, μ, η) in $((\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg}, \wedge)$, i.e.,

$$\mu : \mathcal{T}(H \wedge H) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(H)$$

is a morphism of right $\mathcal{A}$-modules that is associative and unital with respect to

$$\eta : \mathcal{T}(R) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(H).$$

The category of pseudo Hopf algebras and of morphism in $((\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg}, \wedge)$ respecting their multiplicative structure is denoted **PsHopf**.

Remark 2.36. In particular, if (H, μ, η) is a pseudo Hopf algebra, then H is, of course, a chain coalgebra and $\mu_0 := \mu(- \otimes z_0) : H \otimes H \to H$ endows H with an associative multiplication that is a DCSH-map.

Remark 2.37. The embedding $\mathcal{A}$-**Coalg** $\hookrightarrow (\mathcal{A}, \mathcal{F})$-**Coalg** induces an embedding of the category **H** of chain Hopf algebras into the category of pseudo Hopf algebras, i.e., any chain Hopf algebra can be seen as a pseudo Hopf algebra, where the multiplication is a strict coalgebra map.

Definition 2.38. Let H and H' be chain Hopf algebras. A chain map $f : H \to H'$ is a *multiplicative DCSH-map* if there is a morphism $\varphi : \mathcal{T}(H) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(H')$ of pseudo Hopf algebras such that $f = \varphi(- \otimes z_0)$.

Exercise 2.39. A DCSH-map $f : H \to H'$ with corresponding morphism $\varphi : \mathcal{T}(H) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(H')$ of right $\mathcal{A}$-modules is multiplicative if

$$\varphi_{n+1}(xy) = \sum_{\substack{1 \le k \le n+1 \\ \vec{\imath} \in I_{k,n+1}}} \Big(\delta_{H'}^{(i_1)} \otimes \cdots \otimes \delta_{H'}^{(i_k)}\Big)\, \varphi_k(x) * (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})\, \delta_H^{(k)}(y)$$

for all $n \ge 0$ and all $x, y \in H$, where $*$ denotes multiplication in $(H')^{\otimes n+1}$ and $\varphi_k = \varphi(- \otimes z_k)$.

Definition 2.40. Let H be a chain Hopf algebra. A *right H-module coalgebra* is a chain complex M that is both an H-module and a coalgebra, where the H-action map $M \otimes H \to M$ is a map of coalgebras.

Generalize to modules over pseudo Hopf algebras...

Remark 2.41. Given a pseudo Hopf algebra (H, μ, η) in $((\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg}, \wedge)$, a right H-module in $(\mathcal{A}, \mathcal{F})$-**Coalg** consists of a chain coalgebra M, together with a morphism of right $\mathcal{A}$-modules

$$\rho : \mathcal{T}(M \wedge H) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(M)$$

so that $\rho(\rho \wedge H) = \rho(M \wedge \mu)$ and $\rho(M \wedge \eta) = M$.

Remark 2.42. In particular, if (M, ρ) is a right H-module in $(\mathcal{A}, \mathcal{F})$-**Coalg**, then $\rho_0 := \rho(- \otimes z_0) : M \otimes H \to H$ endows M with a right H-action (in **Ch**) that is a DCSH-map.

Remark 2.43. As a consequence of the existence of the embedding $\mathcal{A}$-**Coalg** $\hookrightarrow$ $(\mathcal{A}, \mathcal{F})$-**Coalg**, any H-module coalgebra can be seen as a module over a strict pseudo Hopf algebra, where the action map is a strict coalgebra map.

Definition 2.44. Let $f : H \to H'$ be a multiplicative DCSH-map, with corresponding morphism $\varphi : \mathcal{T}(H) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(H')$ of right $\mathcal{A}$-modules. Let M be a right H-module coalgebra, and let M' be a right H'-comodule algebra. A chain map $g : M \to M'$ is a *DCSH-module map* if there is a morphism $\omega : \mathcal{T}(M) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(M')$ of modules over pseudo Hopf algebras such that $g = \omega(- \otimes z_0)$.

Exercise 2.45. Let $f : H \to H'$ be a multiplicative DCSH-map, with corresponding morphism $\varphi : \mathcal{T}(H) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(H')$ of right $\mathcal{A}$-modules. Let M be a right H-module coalgebra, and let M' be a right H'-comodule algebra. A DCSH map $g : M \to M'$ with corresponding morphism $\omega : \mathcal{T}(M) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}(M')$ of right $\mathcal{A}$-modules is a *DCSH-module map* with respect to φ if

$$\omega_{n+1}(x \bullet y) = \sum_{\substack{1 \le k \le n+1 \\ \vec{\imath} \in I_{k,n+1}}} \left(\delta_{M'}^{(i_1)} \otimes \cdots \otimes \delta_{M'}^{(i_k)}\right) \omega_k(x) \bullet (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})\, \delta_H^{(k)}(y)$$

for all $n \geq 0$ and all $x \in M$ and $y \in H$, where $\bullet$ denotes the action of H on M and the action of $(H')^{\otimes n+1}$ on $(M')^{\otimes n+1}$.

The following theorem (stated slightly imprecisely here, to avoid technical unpleasantness that obscurs the point) is important for enabling inductive construction of pseudo Hopf morphisms between Hopf algebras.

Theorem 2.46. *If H and H' are chain Hopf algebras such that H is free as an algebra on a free graded R-module V with basis B, then any morphism $\theta \in$* $\mathbf{PsHopf}(H, H')$ *is completely determined by the set*

$$\{\theta(v \otimes z_k) \in (H')^{\otimes k+1} \mid v \in \mathsf{B}, k \geq 0\}.$$

2.4. **Alexander-Whitney coalgebras.**

Lemma 2.47. *The induction functor* $\mathrm{Ind} : (\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg} \to \mathcal{A}\text{-}\mathbf{Alg}$ *is comonoidal, i.e., there is a natural transformation of functors into associative chain algebras*

$$q : \mathrm{Ind}(- \wedge -) \to \mathrm{Ind}(-) \otimes \mathrm{Ind}(-),$$

which is given by the Milgram equivalence (Definition 1.24) on objects.

Recall that z_k is the generator of $\mathcal{F}$ in level $k+1$, which is of degree k.

Definition 2.48. The objects of the *weak Alexander-Whitney category* $\mathbf{wF}$ are pairs (C, Ψ), where C is a object in $\mathcal{A}$-**Coalg** and $\Psi \in (\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg}(C, C \otimes C)$ such that

$$\Psi(- \otimes z_0) : C \to C \otimes C$$

is exactly the comultiplication on C, while

$$\mathbf{wF}\big((C, \Psi), (C', \Psi')\big) = \{\theta \in (\mathcal{A}, \mathcal{F})\text{-}\mathbf{Coalg}(C, C') \mid \Psi'\theta = (\theta \wedge \theta)\Psi\}.$$

The objects of $\mathbf{wF}$ are called *weak Alexander-Whitney coalgebras.*

Remark 2.49. Loosely speaking, Alexander-Whitney coalgebras are chain coalgebras such that the comultiplication is a DCSH-map.

Definition 2.50. The objects of the *weak Hopf algebra category* $\mathbf{wH}$ are pairs (A, ψ), where A is a chain algebra over R and $\psi : A \to A \otimes A$ is a map of chain algebras, while

$$\mathbf{wH}\big((A, \psi), (A', \psi')\big) = \{f \in \mathcal{A}\text{-}\mathbf{Alg}(A, A') \mid \psi' f = (f \otimes f)\psi\}.$$

Lemma 2.51. *The cobar construction extends to a functor*

$$\widetilde{\Omega} : \mathbf{wF} \to \mathbf{wH},$$

given by $\widetilde{\Omega}(C, \Psi) = \big(\Omega C, q\, \mathrm{Ind}(\Psi)\big)$, *where*

- $\mathrm{Ind}(\Psi) : \Omega C \to \Omega(C \otimes C)$, *as in Theorem 2.27,*
- $q : \Omega(C \otimes C) \to \Omega C \otimes \Omega C$ *is Milgram's equivalence and*
- $\widetilde{\Omega}\theta = \mathrm{Ind}(\theta) : \Omega C \to \Omega C'$ *for all* $\theta \in \mathbf{wF}\big((C, \Psi), (C', \Psi')\big)$.

Motivated by topology, we are particularly interested in those objects (C, Ψ) of $\mathbf{wF}$ for which $\widetilde{\Omega}(C, \Psi)$ is actually a strict Hopf algebra, i.e., such that $q\,\mathrm{Ind}(\Psi)$ is coassociative.

Definition 2.52. The *Alexander-Whitney category* $\mathbf{F}$ is the full subcategory of $\mathbf{wF}$ such that (C, Ψ) is an object of $\mathbf{F}$ if and only if $q\,\mathrm{Ind}(\Psi)$ is coassociative. We call the objects of $\mathbf{F}$ *Alexander-Whitney coalgebras.*

Remark 2.53. It is clear that $\widetilde{\Omega}$ restricts to a functor

$$\widetilde{\Omega} : \mathbf{F} \to \mathbf{H} \tag{2.5}$$

where $\mathbf{H}$ is the usual category of chain Hopf algebras.

3. Applications of the operadic description of **DCSH**

3.1. **The canonical Adams-Hilton model and chain-level Bott-Samelson.** Theorem 1.33 and Remark 2.53 together imply that there is a sequence of functors

$$\mathbf{sSet}_0 \xrightarrow{\widetilde{C}} \mathbf{F} \xrightarrow{\widetilde{\Omega}} \mathbf{H},$$

with $\widetilde{\Omega}\widetilde{C}(K) = (\Omega C_*K, \psi_K)$, where ψ_K is the Alexander-Whitney diagonal.

Theorem 3.1. *The Szczarba equivalence of chain algebras $Sz_K : \Omega C_*K \to C_*GK$ is a multiplicative DCSH map, with respect to the Alexander-Whitney diagonal on $\Omega C(K)$ and the usual comultiplication on C_*GK.*

Proof. Applying Theorem 2.46, we see that it is enough to construct a family

$$\{\theta(x \otimes z_k) \mid k \geq 0, x \in K\}$$

satisfying appropriate relations with the differentials and comultiplications. We can obtain such a family by an inductive acyclic models argument, using the models described below.

Let $\overline{\Delta}[n]$ denote the quotient of the standard simplicial n-simplex $\Delta[n]$ by its 0-skeleton. Morace and Prouté showed that there is a contracting chain homotopy $\bar{h} : C(G\overline{\Delta}[n]) \to C(G\overline{\Delta}[n])$ in positive degrees. The functor $C\big(G(-)\big)$ from reduced simplicial sets to connected chain algebras is therefore acyclic in positive degrees on the set of models $\mathfrak{M} = \{\overline{\Delta}[n] \mid n \geq 0\}$.

On the other hand, the functor C_* from reduced simplicial sets to connected chain coalgebras is (a retract of something) free on $\mathfrak{M}$. In particular, the set $\{\iota_n \in C(\overline{\Delta}[n]) \mid n \geq 0\}$ gives rise to basis of $C_*(K)$ for all K, where ι_n denotes the unique nondegenerate n-simplex of $\overline{\Delta}[n]$. □

Restricting to suspensions, we can tighten up this result considerably...

Notation 3.2. The simplicial suspension of a simplicial set K is denoted EK and is itself a reduced simplicial set.

Proposition 3.3. *There is a map of chain coalgebras $\alpha : C_*K \to \Omega C_*EK$, where C_*K is endowed with its usual comultiplication Δ_K and ΩC_*EK with the Alexander-Whitney diagonal.*

Proof. Some unpleasant simplicial computations. □

Terminology 3.4. α is the *chain-level James map.*

Theorem 3.5. *Let C_+K denote the kernel of the augmentation map $C_*K \to R$. The chain-level James map induces a natural isomorphism of chain Hopf algebras*

$$\widehat{\alpha} : \big(TC_+K, \hat{d}, \widehat{\Delta}K\big) \xrightarrow{\cong} \big(\Omega C_*EK, \psi_K\big),$$

where $\hat{d}$ and $\widehat{\Delta}_K$ denote, respectively, the derivation determined by the differential d and the algebra map determined by Δ_K.

Proof. Essentially a consequence of the preceeding Proposition. □

Theorem 3.6. *The Szczarba map $Sz_{EK} : \big(\Omega C_*EK, \psi_K\big) \to \big(C_*GEK, \Delta_{GEK}\big)$ is a strict map of chain Hopf algebras for all K.*

Proof. Yet more unpleasant simplicial computations. □

Remark 3.7. These two theorems together imply a *chain-level Bott-Samelson theorem*, i.e., that there is a natural quasi-isomorphism of chain Hopf algebras

$$\big(T\overline{C}_*K, \hat{d}, \widehat{\Delta}_K\big) \xrightarrow{\simeq} C_*GEK$$

for all K.

3.2. **Free loop spaces and (co)Hochschild complexes.** Recall the bar construction and the Hochschild complex on a chain algebra.

Definition 3.8. Let $\mathscr{B}$ denote the *bar construction* functor defined by

$$\mathscr{B}(A,d) = \big(T(s\overline{A}), d_{\mathscr{B}}\big)$$

where

$$\begin{aligned} d_{\mathscr{B}}(sa_1|\cdots|sa_n) = &\sum_{1\leq j\leq n} \pm sa_1|\cdots|s(da_j)|\cdots|sa_n \\ &+ \sum_{1\leq j<n} \pm sa_1|...|s(a_ja_{j+1})|\cdots|sa_n. \end{aligned}$$

Remark 3.9. The graded R-module underlying $\mathscr{B}(A,d)$ is naturally a cofree coassociative coalgebra, with comultiplication given by splitting of words. The differential $d_{\mathscr{B}}$ is a coderivation with respect to this splitting comultiplication, so that $\mathscr{B}(A,d)$ is itself a chain coalgebra. Any chain coalgebra map $\gamma : C \to \mathscr{B}(A,d)$ is determined by its projection to the coalgebra cogenerators $s\overline{A}$, denoted γ_1, which is equivalent to specifying a twisting cochain $t : C \to A$.

Definition 3.10. Let (A,d) be an augmented, associative chain algebra. Let $\mathscr{H}$ denote the *Hochschild complex* functor defined by

$$\mathscr{H}(A,d) = \big(T(s\overline{A}) \otimes A, d_{\mathscr{H}}\big)$$

where

$$\begin{aligned} d_{\mathscr{H}}(sa_1|\cdots|sa_n \otimes b) = &d_{\mathscr{B}}(sa_1|\cdots|sa_n) \otimes b \ \pm sa_1|\cdots|sa_n \otimes db \\ &+ sa_1|\cdots|sa_{n-1} \otimes a_nb \ \pm sa_2|\cdots|sa_n \otimes ba_1. \end{aligned}$$

Remark 3.11. Note that $\mathscr{B}(A,d)$ is a quotient complex of $\mathscr{H}(A,d)$.

There is an interesting and useful dual to the Hochschild complex for a chain coalgebra, extending the cobar construction.

Definition 3.12. Let $\widehat{\mathscr{H}}$ denote the *coHochschild complex* functor defined by

$$\widehat{\mathscr{H}}(C,d) = \big(C \otimes T(s^{-1}\overline{C}), d_{\widehat{\mathscr{H}}}\big)$$

where

$$\begin{aligned} d_{\widehat{\mathscr{H}}}(e \otimes s^{-1}c_1|\cdots|s^{-1}c_n) = &de \otimes s^{-1}c_1|\cdots|s^{-1}c_n \ \pm e \otimes d_\Omega(s^{-1}c_1|\cdots|s^{-1}c_n) \\ &\pm e_i \otimes s^{-1}e^i|s^{-1}c_1|\cdots|s^{-1}c_n \\ &\pm e^i \otimes s^{-1}c_1|\cdots|s^{-1}c_n|s^{-1}e_i, \end{aligned}$$

where the comultiplication applied to e is $e_i \otimes e^i$ (with the convention that applying s^{-1} to an element of degree 0 gives 0).

Remark 3.13. Note that $\Omega(C,d)$ is a subcomplex of $\widehat{\mathscr{H}}(C,d)$.

Recall that $\theta_t : \Omega C \to A$ denotes the chain algebra map induced by a twisting cochain $t : C \to A$. Let $\omega_t : C \to \mathscr{B}A$ denote the chain coalgebra map induced by t.

Proposition 3.14. *A twisting cochain $t : C \to A$ induces a chain map*

$$\omega_t \otimes \theta_t : \widehat{\mathscr{H}}(C, d) \to \mathscr{H}(A, d),$$

which is a quasi-isomorphism if ω_t and θ_t are quasi-isomorphisms.

Proof. Follow your nose through the calculations. □

Application of coHochschild complex to calculations of free loop space homology...

Recall that a simplicial set K is *n-reduced* if K_0 consists of a unique 0-simplex, while K_k contains no nondegenerate simplex for all $1 \leq k \leq n$.

Definition 3.15. Let K be a reduced simplicial set, and let $\mathcal{F}$ denote the free group functor. The *Kan loop group* GK on K is the simplicial group such that $(GK)_n = \mathcal{F}(K_{n+1} \smallsetminus \operatorname{Im} s_0)$, with faces and degeneracies specified by

$$\begin{aligned} \partial_0 \bar{x} &= (\ \overline{\partial_0 x}\ )^{-1} \overline{\partial_1 x}, \\ \partial_i \bar{x} &= \overline{\partial_{i+1} x} \quad \text{for all } i > 0, \\ s_i \bar{x} &= \overline{s_{i+1} x} \quad \text{for all } i \geq 0, \end{aligned}$$

where $\bar{x}$ denotes the class in $(GK)_n$ of $x \in K_{n+1}$.

Definition 3.16. Let K be a simplicial set and G a simplicial group, where the neutral element in any dimension is noted e. A degree -1 map of graded sets $\tau : K \to G$ is a *twisting function* if

$$\begin{aligned} \partial_0 \tau(x) &= \big(\tau(\partial_0 x)\big)^{-1} \tau(\partial_1 x) \\ \partial_i \tau(x) &= \tau(\partial_{i+1} x) \quad i > 0 \\ s_i \tau(x) &= \tau(s_{i+1} x) \quad i \geq 0 \\ \tau(s_0 x) &= e \end{aligned}$$

for all $x \in K$.

Remark 3.17. Let K be a reduced simplicial set.

(1) There is a universal, canonical twisting function $\tau_K : K \to GK$, defined by $\tau_K(x) = \bar{x}$ for all simplices x of K.
(2) Let G be a simplicial group. The set of twisting functions $K \to G$ is in bijective correspondence with the set of morphisms of simplicial groups $GK \to G$. Given a morphism of simplicial groups $h : GK \to G$, the corresponding twisting function is $h\tau_K$.

Definition 3.18. Let $\tau : K \to G$ be a twisting functions. If G operates on the left on a simplicial set L, then the *twisted cartesian product* of K and L, denoted $K \times_\tau L$ is the simplicial set such that $(K \times_\tau L)_n = K_n \times L_n$, with faces and degeneracies given by

$$\begin{aligned} \partial_0(x, y) &= (\partial_0 x, \tau(x) \cdot \partial_0 y) \\ \partial_i(x, y) &= (\partial_i x, \partial_i y) \quad i > 0 \\ s_i(x, y) &= (s_i x, s_i y) \quad i \geq 0. \end{aligned}$$

Remark 3.19. If L is a Kan complex, then the projection $K \times_\tau L \to K$ is a Kan fibration.

Definition 3.20. The *canonical free loop construction* on K, denoted $\mathcal{L}K$, is the twisted cartesian product $K \underset{\tau}{\times} GK$, where $\tau = (\tau_K, \tau_K) : K \to GK \times GK$, and $GK \times GK$ acts on GK by $(v, w) \cdot u = uvw^{-1}$.

Theorem 3.21. *There is a commutative diagram of simplicial sets*

$$\begin{tikzcd} GK \arrow[r, "j"] \arrow[d, "\simeq"] & \mathcal{L}K \arrow[r, "q"] \arrow[d, "\simeq"] & K \arrow[d, "\simeq"] \\ \mathcal{S}_\bullet|GK| \simeq \mathcal{S}_\bullet\Omega|K| \arrow[r, "\mathcal{S}_\bullet i"] & \mathcal{S}_\bullet\Lambda|K| \arrow[r, "\mathcal{S}_\bullet e"] & \mathcal{S}_\bullet|K|, \end{tikzcd}$$

where j and q are the obvious inclusion and projection and

$$\Omega|K| \xrightarrow{i} \Lambda|K| \xrightarrow{e} |K|$$

is the usual free loop fibration sequence.

The classical theorem of E. Brown, R. Brown and Gugenheim clarifying the relation between twisting functions and twisting cochains...

Theorem 3.22. *For each twisting function $\tau : K \to G$ and every simplicial set L admitting a left action by G, there exists a twisting cochain $t(\tau) : C_*K \to C_*G$ and an SDR*

$$C_*K \otimes_{t(\tau)} C_*L \underset{f_\tau}{\overset{\nabla_\tau}{\rightleftarrows}} C_*(K \times_\tau L) \circlearrowleft \varphi_\tau,$$

where $t(\tau)$, ∇_τ, f_τ and φ_τ can be chosen naturally.

Applying Theorem 3.22 to the twisting function $\tau = (\tau_K, \tau_K) : K \to GK \times GK$ and to the conjugation action of $GK \times GK$ on GK, we obtain an SDR

$$C_*K \otimes_{t(\tau)} C_*GK \underset{f_\tau}{\overset{\nabla_\tau}{\rightleftarrows}} C_*(\mathcal{L}K) \circlearrowleft \varphi_\tau.$$

Theorem 3.23. *For any reduced simplicial set K, there is a commutative diagram of chain complexes*

(3.1)
$$\begin{tikzcd} \Omega C_*K \arrow[r] \arrow[d, "Sz_K"', "\simeq"] & \widehat{\mathscr{H}}(C_*K) \arrow[r, two heads] \arrow[d, "\widetilde{Sz}_K"', "\simeq"] & C_*K \arrow[d, equal] \\ C_*GK \arrow[r] \arrow[dr, "C_*j"'] & C_*K \otimes_{t(\tau)} C_*GK \arrow[r, two heads] \arrow[d, "\nabla_\tau"', "\simeq"] & C_*K \\ & C_*\mathcal{L}K \arrow[ur, "C_*q"'] & \end{tikzcd}$$

where $\widetilde{Sz}_K$ is a morphism of ΩC_*K-modules, with respect to Sz_K.

Proof. Acyclic models, using $\{\overline{\Delta}[n] \mid n \geq 0\}$ again. □

3.3. Multiplicative structures in equivariant cohomology.

3.3.1. *Linear structure of equivariant homology.* In this section, let C_*X denote singular or cubical chains on a topological spaces X. Let E be the total space of a principal G-bundle, where G is a connected topological group. Let Y be any G-space. The multiplication map $\mu : G \times G \to G$ induces the structure of a chain algebra on C_*G, with multiplication map given by the composite

$$C_*G \otimes C_*G \xrightarrow{EZ} C_*(G \times G) \xrightarrow{C_*\mu} C_*G,$$

where EZ is the natural Eilenberg-Zilber equivalence. The action maps $E \times G \to E$ and $G \times Y \to Y$ similarly induce C_*G-module structures on C_*E and on C_*Y.

Theorem 3.24 (Moore)**.** *There is an isomorphism of graded $\mathbb{Z}$-modules*

$$H_*(E \underset{G}{\times} Y) \cong \operatorname{Tor}_*^{C_*G}(C_*E, C_*Y).$$

The goal of this lecture is to explain how to enrich Moore's theorem, obtaining a comultiplicative isomorphism, by taking into account in a coherent manner the comultiplicative structure on C_*G, C_*E and C_*Y, then to analyze in more detail the special case $G = S^1$ and $E = ES^1$.

3.3.2. *A comultiplicative enrichment of Moore's theorem.* Recall the notions of multiplicative DCSH-map (Definition 2.38) and of DCSH-module map (Definition 2.44).

Remark 3.25. Let H be a chain Hopf algebra. Suppose that M and M' are right and left H-module coalgebras, with structure maps ρ and λ, respectively. Consider the following coequalizer of chain complexes.

$$M \otimes H \otimes M' \underset{M\otimes\lambda}{\overset{\rho\otimes M'}{\rightrightarrows}} M \otimes M' \xrightarrow{\pi} M \otimes_H M'$$

Since $\rho \otimes M'$ and $M \otimes \lambda$ are both maps of coalgebras, $M \underset{H}{\otimes} M'$ admits a coalgebra structure such that the quotient map π is a coalgebra map.

Theorem 3.26. *Let $\theta : H \longrightarrow K$ be a multiplicative DCSH-map. Let M and M' be right and left H-module coalgebras, and let N and N' be right and left K-module coalgebras. Let $\varphi : M \longrightarrow N$ and $\varphi' : M' \longrightarrow N'$ be DCSH-module maps with respect to θ. Then the induced chain map*

$$\varphi \underset{\theta}{\otimes} \varphi' : M \underset{H}{\otimes} M' \longrightarrow N \underset{K}{\otimes} N'$$

is a DCSH-map. Furthermore, if in addition M' and N' are right L-module coalgebras, where L is a chain Hopf algebra, and φ' is a DCSH-module map with respect to Id_L, then $\varphi \underset{\theta}{\otimes} \varphi'$ is a DCSH-module map with respect to Id_L as well.

Proof. Use that colimits in $\mathbf{Ch}^{\Sigma}$ are calculated level-wise and that $- \underset{\mathcal{A}}{\diamond} \mathcal{F}$ preserves colimits. ☐

Definition 3.27. Let $\theta : H \to K$ be a multiplicative DCSH map between chain Hopf algebras. Let $\varphi : M \to N$ be a DCSH-module map with respect to θ, where M is a right H-comodule algebra and N is a right K-comodule algebra. If M is a semifree extension of H and φ is a quasi-isomorphism, then φ is a *DCSH H-resolution* of N.

The enriched version of Moore's theorem can now be stated as follows.

Theorem 3.28. *Given a multiplicative DCSH-quasi-isomorphism $f : H \xrightarrow{\simeq} C_*G$ and a DCSH-H-resolution of C_*E, $g : M \xrightarrow{\simeq} C_*E$, there is a DCSH quasi-isomorphism*

$$M \underset{H}{\otimes} C_*Y \xrightarrow{\simeq} C_*(E \underset{G}{\times} Y).$$

In particular, there is an isomorphism of graded algebras

$$H^*\left((M \underset{H}{\otimes} C_*Y)^\sharp \right) \cong H^*(E \underset{G}{\times} Y),$$

where the superscript $\sharp$ denotes the R-linear dual.

Proof. Recall that Moore proved that given any CU_*G-semifree resolution of CU_*E,

$$\psi : N \xrightarrow{\simeq} CU_*E$$

the composite

$$N \underset{CU_*G}{\otimes} CU_*Y \xrightarrow{\psi\otimes 1} CU_*E \underset{CU_*G}{\otimes} CU_*Y \to CU_*(E \underset{G}{\times} Y)$$

is a quasi-isomorphism.

Observe now that according to Theorem 3.26, the induced map

$$\varphi \underset{\theta}{\wedge} 1_{CU_*G} : M \underset{H}{\otimes} CU_*G \to CU_*E \underset{CU_*G}{\otimes} CU_*G \cong CU_*E$$

is a DCSH-module map with respect to 1_{CU_*G}.

Applying some results of Félix-Halperin-Thomas on semifree extensions, we obtain

$$\begin{tikzcd}
M \underset{H}{\otimes} CU_*Y \arrow[r, "\cong"] & (M \underset{H}{\otimes} CU_*G) \underset{CU_*G}{\otimes} CU_*Y \arrow[d, "(\varphi \underset{\theta}{\wedge} 1_{CU_*G}) \underset{1_{CU_*G}}{\wedge} 1_{CU_*Y}"] \\
 & CU_*E \underset{CU_*G}{\otimes} CU_*Y \arrow[d] \\
 & CU_*(E \underset{G}{\times} Y),
\end{tikzcd}$$

where the first map is an isomorphism of coalgebras and the last map is the strict coalgebra map. According to Theorem 3.26, the first vertical map is a DCSH-map. Finally, by Moore's theorem, the composite of the two vertical maps is a quasi-isomorphism, so we can conclude. □

3.3.3. *Homotopy orbits of circle actions.* Now let C_* refer exclusively to cubical chains.

Identification of a special family of primitives in C_*S^1 is the key to applying Theorem 3.28 to computing S^1-equivariant (co)homology.

Proposition 3.29. *There is a set $\{T_k \in C_{2k+1}S^1 \mid k \geq 0\}$ of primitives such that T_0 represents the generator of H_1S^1 and $dT_k = \sum_{i=0}^{k-1} T_i \cdot T_{k-i-1}$ for all k.*

Recall that $H_*BS^1 \cong R[u_2]$ as graded R-modules.

Theorem 3.30. *There is a quasi-isomorphism of chain Hopf algebras*

$$f : \Omega H_* BS^1 \xrightarrow{\simeq} C_* S^1 : s^{-1}(u^k) \mapsto T_{k-1},$$

extending to a DCSH-module quasi-isomorphism with respect to f:

$$g : H_* BS^1 \otimes_{t_\Omega} \Omega H_* BS^1 \xrightarrow{\simeq} C_* ES^1,$$

where $H_* BS^1 \otimes_{t_\Omega} \Omega H_* BS^1$ denotes the acyclic cobar construction on $H_* BS^1$.

Proof. Let $\mathrm{H}_* BS^1 \otimes_{t_\Omega} \Omega\, \mathrm{H}_* BS^1$ denote the acyclic cobar construction on $\mathrm{H}_* BS^1$. It is easy to see that the usual tensor coproduct commutes with the differential, i.e., $\mathrm{H}_* BS^1 \otimes_{t_\Omega} \Omega\, \mathrm{H}_* BS^1$ is a chain coalgebra, with untwisted coproduct.

Let $j : S^1 \longrightarrow ES^1$ denote the inclusion of S^1 as the base of the construction of ES^1, which is an S^1-equivariant map. The composite

$$C_* j \circ \widehat{\zeta} : \Omega\, \mathrm{H}_* BS^1 \longrightarrow C_* ES^1$$

is map of $\Omega\, \mathrm{H}_* BS^1$-module coalgebras. Consider the following commutative diagram of right $\Omega\, \mathrm{H}_* BS^1$-module coalgebras.

$$\begin{tikzcd}
\Omega\, \mathrm{H}_* BS^1 \arrow[r, "C_* j \circ \widehat{\zeta}"] \arrow[d, "\iota"] & C_* ES^1 \arrow[d, two heads, "\simeq"'] \\
\mathrm{H}_* BS^1 \otimes_{t_\Omega} \Omega\, \mathrm{H}_* BS^1 \arrow[r, "\simeq"] & R
\end{tikzcd}$$

Since the inclusion ι is a coalgebra map, and therefore a DCSH-map, and the composite $C_* j \circ \widehat{\zeta}$ is a $\Omega\, \mathrm{H}_* BS^1$-module coalgebra map, and therefore a DCSH-module map, the operadic description of **DCSH** implies that they can be realized as morphisms of right $\mathcal{A}$-modules, as noted below.

$$\begin{tikzcd}
\mathcal{T}\big(\Omega\, \mathrm{H}_* BS^1\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} \arrow[r, "\mathfrak{j}"] \arrow[d, "\mathcal{T}(\iota) \underset{\mathcal{A}}{\diamond} 1"'] & \mathcal{T}\big(C_* ES^1\big) \\
\mathcal{T}\big(\mathrm{H}_* BS^1 \otimes_{t_\Omega} \Omega\, \mathrm{H}_* BS^1\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} &
\end{tikzcd}$$

Since $C_* ES^1$ is acyclic, $\mathrm{H}_* BS^1 \otimes_{t_\Omega} \Omega\, \mathrm{H}_* BS^1$ is semifree, the underlying algebra of $\Omega\, \mathrm{H}_* BS^1$ is free and $\mathcal{F}$ is filtered, we can inductively construct an extension $\mathfrak{k}$ of $\mathfrak{j}$ so that

$$\begin{tikzcd}
\mathcal{T}\big(\Omega\, \mathrm{H}_* BS^1\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} \arrow[r, "\mathfrak{j}"] \arrow[d, "\mathcal{T}(\iota) \underset{\mathcal{A}}{\diamond} 1"'] & \mathcal{T}\big(C_* ES^1\big) \\
\mathcal{T}\big(\mathrm{H}_* BS^1 \otimes_{t_\Omega} \Omega\, \mathrm{H}_* BS^1\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} \arrow[ur, "\mathfrak{k}"'] &
\end{tikzcd}$$

commutes. Furthermore, $\mathfrak{k}$ is necessarily a quasi-isomorphism, since both $CU_* ES^1$ and $\mathrm{H}_* BS^1 \otimes_{t_\Omega} \Omega\, \mathrm{H}_* BS^1$ are acyclic. In other words, restricting to level 1, we obtain a DCSH $\Omega\, \mathrm{H}_* BS^1$-resolution of $CU_* ES^1$:

$$\mathrm{H}_* BS^1 \otimes_{t_\Omega} \Omega\, \mathrm{H}_* BS^1 \xrightarrow{\simeq} C_* ES^1. \tag{3.2}$$

□

Remark 3.31. Since $H_*BS^1 \otimes_{t_\Omega} \Omega H_*BS^1$ is ΩH_*BS^1-semifree, g is a DCSH ΩH_*BS^1-resolution of C_*ES^1.

Corollary 3.32. *Let Y be a left S^1-space. There is a DCSH-quasi-isomorphism*

$$\big(H_*BS^1 \otimes_{t_\Omega} \Omega H_*BS^1\big) \underset{\Omega H_*BS^1}{\otimes} C_*Y \xrightarrow{\simeq} C_*(ES^1 \underset{S^1}{\times} Y) = C_*Y_{hS^1},$$

which gives rise upon dualization to a strongly homotopy-multiplicative quasi-isomorphism

$$(R[u] \otimes C^*Y, D) \simeq\longrightarrow C^*Y_{hS^1}\ .$$

In particular, $\mathrm{H}^*(R[u] \otimes C^*Y, D) \cong \mathrm{H}^*\, Y_{hS^1}$ *as graded algebras. Here, for all $y \in C^*Y$, $D(u^n \otimes y) = u^n \otimes y + \sum_{k\geq 0} u^{n+k+1} \otimes \omega_k(y)$, where $\omega_k : C^*Y \to C^{*-2k-1}Y$ is a derivation such that $[d, \omega_k] = -\sum_{i=0}^{k-1} \omega_i \circ \omega_{k-i-1}$. In particular, $\omega_0 : C^*Y \to C^{*-1}Y$ is a chain map inducing the Δ-operation of the Batalin-Vilkoviskiy structure on H^*Y.*

4. Cobar constructions in monoidal model categories

4.1. Monoidal categories and model categories.

Definition 4.1. A *monoidal category* $(\mathbf{C}, \otimes, I)$ consists of a category $\mathbf{C}$, endowed with a bifunctor

$$-\otimes - : \mathbf{C} \times \mathbf{C} \to \mathbf{C},$$

called the *monoidal product*, and a distinguished object I, called the *unit object* such that the monoidal product is associative and unital up to natural isomorphism. In other words, for all $A \in \operatorname{Ob}\mathbf{C}$, there are natural isomorphisms $A \otimes I \cong A \cong I \otimes A$ and for all $A, B, C \in \operatorname{Ob}\mathbf{C}$, there are natural isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. Furthermore, the natural isomorphisms must be appropriately coherent. The monoidal category $(\mathbf{C}, \otimes, I)$ is *symmetric* if there is a natural isomorphism $A \otimes B \cong B \otimes A$ for all $A, B \in \operatorname{Ob}\mathbf{C}$, which is coherent with the previous natural isomorphisms.

Definition 4.2. Let $(\mathbf{C}, \otimes, I)$ be a monoidal category. A *comonoid* in $\mathbf{C}$ is an object C in $\mathbf{C}$, together with two morphisms in $\mathbf{C}$: a comultiplication map $\Delta : C \to C \otimes C$ and a counit map $\varepsilon : C \to I$ such that Δ is coassociative and counital, i.e., the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes C"] \\ C \otimes C \arrow[r, "C\otimes\Delta"] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\cong"'] & C \otimes C \arrow[dl, "C\otimes\varepsilon"] \arrow[dr, "\varepsilon\otimes C"'] & C \arrow[l, "\Delta"'] \arrow[d, "\cong"] \\ C \otimes I & & I \otimes C \end{tikzcd}$$

must commute, where the isomorphisms are the natural isomorphisms mentioned above.

Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be comonoids in a monoidal category $(\mathbf{C}, \otimes, I)$. A *morphism of comonoids* from (C, Δ, ε) to $(C', \Delta', \varepsilon')$ is a morphism $f \in \mathbf{C}(C, C')$ such that the diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes C \arrow[r, "f\otimes f"] & C' \otimes C' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & C'' \arrow[dl, "\varepsilon'"] \\ & I & \end{tikzcd}$$

commute.

There are obvious dual notions of *monoid* and of *morphism of monoids* in any monoidal category.

Notation 4.3. We often abuse terminology slightly and refer to a (co)monoid simply by its underlying object in the category $\mathbf{C}$, just as we sometimes write only the underlying category when naming a monoidal category.

Definition 4.4. Let (C, Δ, ε) be a comonoid in a monoidal category $(\mathbf{C}, \otimes, I)$. A *right C-comodule* in $\mathbf{C}$ is an object M in $\mathbf{C}$ together with a morphism $\rho : M \to M \otimes C$ in $\mathbf{C}$, called the *coaction map*, such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "\rho\otimes C"] \\ M \otimes C \arrow[r, "M\otimes\Delta"] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\cong"'] & M \otimes C \arrow[dl, "M\otimes\varepsilon"] \\ M \otimes I & \end{tikzcd}$$

commute, where the isomorphism is the natural isomorphism of the monoidal structure on $\mathbf{C}$.

Let (M, ρ) and (M', ρ') be right C-comodules. A *morphism of right C-comodules* from (M, ρ) to (M', ρ') is a morphism $g \in \mathbf{C}(M, M')$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\rho"'] & M' \arrow[d, "\rho'"] \\ M \otimes C \arrow[r, "g\otimes C"] & M' \otimes C \end{tikzcd} \tag{4.1}$$

commutes. The category of right C-comodules and their morphisms is denoted $\mathbf{Comod}_C$.

Remark 4.5. The forgetful functor $U : \mathbf{Comod}_C \to \mathbf{C}$ admits a right adjoint $F : \mathbf{C} \to \mathbf{Comod}_C$ where $F(X) = X \otimes C$, with action map given by

$$X \otimes \Delta : X \otimes C \to X \otimes C \otimes C.$$

We call $X \otimes C$ the *cofree left C-comodule* generated by X.

Exercise 4.6. If $\rho : M \to M \otimes C$ is a right coaction, then ρ is a morphism of right C-comodules, with respect to the cofree coaction on $M \otimes C$.

The category ${}_C\mathbf{Comod}$ of left comodules over a comonoid C and their morphisms is defined analogously, in terms of coaction maps $\lambda : M \to C \otimes M$. For any object X of $\mathbf{C}$, the *cofree left C-module* generated by X is $C \otimes X$, endowed with the action map $\Delta \otimes X : C \otimes X \to C \otimes C \otimes X$.

Definition 4.7. If an object M in $\mathbf{C}$ is endowed with compatible left and right action maps $\lambda : M \to C \otimes M$ and $\rho : M \to M \otimes C$, then (M, λ, ρ) is an *C-bicomodule*. Here, compatibility means that the diagram

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\lambda"'] & M \otimes C \arrow[d, "\lambda\otimes C"] \\ C \otimes M \arrow[r, "C\otimes\rho"] & C \otimes M \otimes C \end{tikzcd}$$

commutes. A *morphism of C-bicomodules* from (M, λ, ρ) to (M', λ', ρ') is a morphism $g \in \mathbf{C}(M, M')$ that is both a morphism of left C-comodules and a morphism of right C-comodules. The category of C-bicomodules and their morphisms is denoted ${}_C\mathbf{Comod}_C$. The *cofree C-bicomodule* generated by an object X of $\mathbf{C}$ is $C \otimes X \otimes C$, with the usual left and right C-coaction maps.

Definition 4.8. Let $\mathbf{C}$ be a monoidal category admitting equalizers. Let (M, ρ) and (N, λ) be a right and a left C-comodule, respectively. The *cotensor product* $M \underset{C}{\square} N$ of M and N is the equalizer

$$M \underset{C}{\square} N \to M \otimes N \underset{\rho \otimes Id_N}{\overset{M \otimes \lambda}{\rightrightarrows}} M \otimes C \otimes N,$$

which is computed in $\mathbf{C}$. Since this construction is clearly natural in M and in N, there is in fact a bifunctor

$$- \underset{C}{\square} - : \mathbf{Comod}_C \times {}_C\mathbf{Comod} \to \mathbf{C}.$$

Remark 4.9. If $N = I$ with its usual left C-coaction, then

$$M \underset{C}{\square} I = \text{equal}(M \underset{\rho}{\overset{M \otimes \eta}{\rightrightarrows}} M \otimes C).$$

In other words $M \underset{C}{\square} I$ can be seen as the *fixed points* of the coaction ρ, justifying the notation $M^C := M \underset{C}{\square} I$ that we use henceforth. A similar observation applies to $N^C := I \underset{C}{\square} N$ for all $(N, \lambda) \in {}_C\mathbf{Comod}$.

Definition 4.10. A *model category* consists of a category $\mathbf{C}$, together with classes of morphisms $\mathsf{WE}, \mathsf{Fib}, \mathsf{Cof} \subset \operatorname{Mor} \mathbf{C}$ that are closed under composition and contain all identities, such that the following axioms are satisfied.

(M1) All finite limits and colimits exist.

(M2) Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be morphisms in $\mathbf{C}$. If two of f, g, and gf are in WE, then so is the third.

(M3) The classes WE, Fib, and Cof are all closed under taking retracts.

(M4) $\mathsf{Cof} \subseteq LLP(\mathsf{Fib} \cap \mathsf{WE})$ and $\mathsf{Fib} \subseteq RLP(\mathsf{Cof} \cap \mathsf{WE})$.

(M5) If $f \in \operatorname{Mor} \mathbf{C}$, then there exist
  (a) $i \in \mathsf{Cof}$ and $p \in \mathsf{Fib} \cap \mathsf{WE}$ such that $f = pi$;
  (b) $j \in \mathsf{Cof} \cap \mathsf{WE}$ and $q \in \mathsf{Fib}$ such that $f = qj$.

By analogy with the homotopy structure in the category of topological spaces, the morphisms belonging to the classes WE, Fib and Cof are called *weak equivalences*, *fibrations*, and *cofibrations* and are denoted by decorated arrows $\xrightarrow{\sim}$, $\twoheadrightarrow$, and $\rightarrowtail$. The elements of the classes $\mathsf{Fib} \cap \mathsf{WE}$ and $\mathsf{Cof} \cap \mathsf{WE}$ are called, respectively, *acyclic fibrations* and *acyclic cofibrations*. Since WE, Fib and Cof are all closed under composition and contain all isomorphisms, we can and sometimes do view them as subcategories of $\mathbf{C}$, rather than simply as classes of morphisms.

Axiom (M1) implies that any model category has an initial object ϕ and a terminal object e. An object A in a model category is *cofibrant* if the unique morphism $\phi \longrightarrow A$ is a cofibration. Similarly, A is *fibrant* if the unique morphism $A \longrightarrow e$ is a fibration.

We sometimes need in this lecture to impose the following additional condition on the model categories we consider.

Definition 4.11. A model category is *right proper* if in every pullback diagram

$$\begin{tikzcd} \bullet \arrow[r, "\bar{f}"] \arrow[d, two heads, "\bar{p}"] & \bullet \arrow[d, two heads, "p"] \\ \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$

the morphism $\bar{f}$ is a weak equivalence if f is a weak equivalence.

When defining a homotopy-invariant replacement for the cotensor product, we need to understand the homotopy theory of diagrams of the form

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in a model category. The following proposition assembles the necessary results on diagram categories, from Hovey's book on model categories, applied to the specific diagram category that interests us.

Theorem 4.12. *Let* $\mathbf{E}$ *denote the category with two objects, denoted* 0 *and* 1*, and two nonidentity morphisms* $s, t : 1 \to 0$*. Let* $\mathbf{C}$ *denote a model category. The functor category* $\mathbf{C}^{\mathbf{E}}$ *admits a model stucture in which weak equivalences and cofibrations are defined objectwise and in which* $\varphi : X \to Y$ *is a fibration if and only if*

(1) $\varphi_0 : X(0) \to Y(0)$ *is a fibration in* $\mathbf{C}$*; and*
(2) *the induced morphism*
$$X(1) \to Y(1) \underset{Y(0)\times Y(0)}{\times} (X(0) \times X(0))$$
is a fibration in $\mathbf{C}$.

In particular, every fibration is an objectwise fibration. Moreover, if $\varphi : X \to Y$ *is a fibration (respectively, acyclic fibration) in* $\mathbf{C}^{\mathbf{E}}$*, then* $\lim \varphi : \lim X \to \lim Y$ *is a fibration (resp., acyclic fibration) in* $\mathbf{C}$.

Corollary 4.13. *If* X *and* Y *are fibrant objects in* $\mathbf{C}^{\mathbf{E}}$ *and* $\varphi : X \to Y$ *is an objectwise weak equivalence, then* $\lim \varphi : \lim X \to \lim Y$ *is a weak equivalence.*

Proof. Apply Ken Brown's Lemma to the previous theorem. □

Remark 4.14. By Theorem 4.12, an object X in $\mathbf{C}^{\mathbf{E}}$ is fibrant if and only if $X(0)$ is fibrant in $\mathbf{C}$ and

$$\big(X(s), X(t)\big) : X(1) \to X(0) \times X(0)$$

is a fibration in $\mathbf{C}$. In particular, $X(s) : X(1) \to X(0)$ and $X(t) : X(1) \to X(0)$ must be fibrations if X is fibrant.

Remark 4.15. Note that if $X : \mathbf{E} \to \mathbf{C}$ is an object in $\mathbf{C}^{\mathbf{E}}$, then $\lim X$ is exactly the equalizer of

$$X(1) \underset{X(t)}{\overset{X(s)}{\rightrightarrows}} X(0),$$

so that the theorem above and its corollary provide conditions under which a weak equivalence of equalizer diagrams induces a weak equivalence of the equalizers.

4.2. **Two-sided cobar constructions.** The notion necessary to the construction of a homotopy-invariant version of the cotensor product....

Definition 4.16. Let (C, Δ, ε) be a comonoid in $\mathbf{C}$. Let (M, ρ) be a right C-comodule, and let (N, λ) be a left C-comodule. A *two-sided cobar construction* on M is a factorization

$$\begin{tikzcd} M \arrow[rr, "\rho"] \arrow[dr, "\sim", "j"'] & & M \otimes C \\ & \Omega(M; C; C) \arrow[ur, "\widehat{\rho}"'] & \end{tikzcd}$$

in $\mathbf{C}$, while a *two-sided cobar construction* on N is a factorization

$$\begin{tikzcd} N \arrow[rr, "\lambda"] \arrow[dr, "\sim", "j"'] & & C \otimes N \\ & \Omega(C; C; N) \arrow[ur, "\widehat{\lambda}"'] & \end{tikzcd}$$

in $\mathbf{C}$. A two-sided cobar construction is *good* if $\widehat{\rho}$, respectively $\widehat{\lambda}$, is a fibration and *very good* if, in addition, j is a cofibration. If j and $\widehat{\rho}$, respectively $\widehat{\lambda}$, are morphisms of C-comodules, then the two-sided cobar construction *respects coactions.*

Exercise 4.17. (Working with field coefficients for the sake of simplicity...) Let $\mathbf{C} = \mathbf{Ch}(\Bbbk)$, the category of chain complexes of $\Bbbk$-vector spaces, endowed with the model category structure in which fibrations are degreewise surjections (in positive degrees), cofibrations are degreewise injections and weak equivalences are quasi-isomorphisms. Both (MM1) and (MM2) are clearly satisfied in this structure, with respect to the usual tensor product of chain complexes. Let (M, ρ) be a right C-comodule, where C is a coaugmented, coassociative chain coalgebra. Then

$$\begin{tikzcd} M \arrow[rr, "\rho"] \arrow[dr, "\sim", "j"'] & & M \otimes C \\ & M \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} C \arrow[ur, "\widehat{\rho}"'] & \end{tikzcd}$$

is a very good two-sided cobar construction that respects coactions, where

$$j(x) = x \otimes 1 \otimes 1 + x_i \otimes 1 \otimes c^i$$

for all $x \in \mathbf{M}$ (with $\rho(x) = x \otimes 1 + x_i \otimes c^i$) and

$$\widehat{\rho} = M \otimes \varepsilon \otimes C.$$

Proposition 4.18. *Two-sided cobar constructions exist for any comonoid C in a cofibrantly generated monoidal model category $\mathbf{C}$. In particular, there are functors*

$$\Omega(-; C; C) : \mathbf{Comod}_C \to \mathbf{C} \quad \textit{and} \quad \Omega(C; C; -) : {}_C\mathbf{Comod} \to \mathbf{C},$$

together with natural acyclic cofibrations

$$Id \overset{\sim}{\rightarrowtail} \Omega(-; C; C) \quad \textit{and} \quad Id \overset{\sim}{\rightarrowtail} \Omega(C; C; -)$$

and natural fibrations

$$\Omega(-; C; C) \twoheadrightarrow - \otimes C \quad \textit{and} \quad \Omega(C; C; -) \twoheadrightarrow C \otimes - \,.$$

Remark 4.19. Unlike the dual situation for modules, the category of comodules over a comonoid in a cofibrantly generated, monoidal model category $\mathbf{C}$ does not inherit a model category structure from $\mathbf{C}$, since the cofree comodule functor is a *right* adjoint. In general we cannot therefore suppose that the two-sided cobar constructions of the proposition above respect coactions.

Definition 4.20. Suppose that C is coaugmented, i.e., endowed with a morphism of comonoids $\eta : I \to C$, where the comultiplication on I is given by the canonical isomorphism $I \cong I \otimes I$. The unit object I is both a left and a right C-comodule, with coaction maps given by the coaugmentation $\eta : I \to C$, followed by the appropriate canonical isomorphism: $C \cong C \otimes I$ or $C \cong I \otimes C$. An *acyclic cobar construction* is then a two-sided cobar construction of the following special form

$$\begin{tikzcd} I \arrow[rr, "\eta"] \arrow[dr, "j"', "\sim"] & & C . \\ & \Omega(C;C) \arrow[ur, "\widehat{\eta}"'] & \end{tikzcd}$$

Remark 4.21. Let

$$\begin{tikzcd} M \arrow[rr, "\rho"] \arrow[dr, "j"', "\sim"] & & M \otimes C \\ & \Omega(M;C;C) \arrow[ur, "\widehat{\rho}"'] & \end{tikzcd}$$

be a two-sided cobar construction on (M, ρ). Since

$$(M \otimes \varepsilon)\widehat{\rho} j = (M \otimes \varepsilon)\rho = M$$

and j is a weak equivalence, by axiom (M2), the retraction

$$r_\rho := (M \otimes \varepsilon)\widehat{\rho} : \Omega(M;C;C) \to M$$

is also a weak equivalence. Similarly, for any two-sided cobar construction

$$\begin{tikzcd} N \arrow[rr, "\lambda"] \arrow[dr, "j"', "\sim"] & & C \otimes N \\ & \Omega(C;C;N) \arrow[ur, "\widehat{\lambda}"'] & \end{tikzcd}$$

j admits a retraction $r_\lambda : \Omega(C;C;N) \to N$, which is a weak equivalence since j is.

We now consider homotopy invariance of two-sided cobar constructions.

Lemma 4.22. *Let* $f : (M, \rho) \xrightarrow{\sim} (M', \rho')$ *be a weak equivalence of right* C*-comodules. For any two-sided cobar constructions*

$$M \underset{\sim}{\overset{j}{\rightarrowtail}} \Omega(M;C;C) \overset{\widehat{\rho}}{\twoheadrightarrow} M \otimes C$$

and

$$M \underset{\sim}{\xrightarrow{j'}} \Omega(M';C;C) \overset{\widehat{\rho}'}{\twoheadrightarrow} M' \otimes C \,,$$

there is a weak equivalence

$$\Omega f : \Omega(M;C;C) \xrightarrow{\sim} \Omega(M';C;C)$$

such that

$$\begin{tikzcd}
M \arrow[rr, "\rho"] \arrow[dr, "j"', "\sim"] \arrow[ddd, "f"', "\sim"] & & M\otimes C \arrow[ddd, "f\otimes C"'] \\
& \Omega(M;C;C) \arrow[ur, "\widehat{\rho}"'] \arrow[d, "\Omega f"', "\sim"] & \\
& \Omega(M';C;C) \arrow[dr, "\widehat{\rho}'"] & \\
M' \arrow[rr, "\rho'"] \arrow[ur, "j'", "\sim"'] & & M'\otimes C
\end{tikzcd}$$

commutes.

A similar result holds for left comodules.

Proof. Apply the second part of axiom (M4) to the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "j'f", "\sim"'] \arrow[d, tail, "j", "\sim"'] & \Omega(M';C;C) \arrow[d, two heads, "\widehat{\rho}'"] \\
\Omega(M;C;C) \arrow[r, "(f\otimes C)\widehat{\rho}"] & M'\otimes C.
\end{tikzcd}$$

$\square$

Remark 4.23. Under the hypotheses of Lemma 4.22, it follows that $r_{\rho'}\Omega f = fr_\rho$, since $(f\otimes C)\widehat{\rho} = \widehat{\rho}'\Omega f$ and $(M'\otimes\varepsilon)(f\otimes C) = f(M\otimes\varepsilon)$.

4.3. **The homotopy cotensor product.** We now introduce a homotopy-invariant version of the cotensor product.

Motivation 4.24. There are three primary motivations for the introduction of a homotopy cotensor product.

(1) To provide a deep explanation of the Eilenberg-Moore theorems relating homology of topological pullbacks to certain Cotors (see section 4.6), leading to possible generalizations of these theorems.
(2) To develop a theory of homotopic Hopf-Galois extensions in monoidal model categories, generalizing Rognes' (Hopf-)Galois theory for ring spectra, and then
(3) To develop a deeper understanding of the slogan "principal fibrations are Galois extensions".

To explain why homotopy cotensor products are necessary to goal (2) and thus to goal (3), I outline briefly the notion of a Hopf-Galois extension in a (symmetric) monoidal model category $\mathbf{C}$. Let $\varphi : A \to B$ be a morphism of commutative monoids in $\mathbf{C}$, so that B can be seen as an A-algebra (i.e., a monoid in the category of A-bimodules) via φ. Let H be a Hopf monoid (i.e., a monoid with compatible comonoid structure) in $\mathbf{C}$, which we consider as an A-algebra with trivial A-module structure. Suppose that B is a right H-comonoid in the category of A-bimodules, i.e., there is a coaction map $\rho : B \to B\otimes H$ in the category of A-bimodules.

Generalizing Rognes' definition, we say that $\varphi : A \to B$ is an *H-Hopf-Galois extension* if

- the natural induced map $A \to B^{hH}$ is a weak equivalence, where B^{hH} denotes the *homotopy cofixed points* of the coaction ρ; and
- the composite
$$B \otimes_A B \xrightarrow{B \otimes_A \rho} B \otimes_A B \otimes H \xrightarrow{\mu \otimes H} B \otimes H$$
is also a weak equivalence.

Thus, to state this definition precisely, we need a meaningful definition of the "homotopy cofixed points" of a coaction in a monoidal model category. By analogy with the constructions already seen for chain complexes, we define "homotopy cofixed points" as a special case of the "homotopy cotensor product."

Notation 4.25. Throughout this section $(\mathbf{C}, \otimes, I)$ denotes a category admitting model and monoidal structures such that

(MM1) the set of weak equivalences with cofibrant domain is preserved under $\otimes$; and

(MM2) the set of fibrations with cofibrant domain is preserved under $\otimes$.

Remark 4.26. Many well-known and frequently used model categories satisfy (MM1) and (MM2). We'll see examples later.

Definition 4.27. Let

$$\begin{tikzcd} M \arrow[rr, "\rho"] \arrow[dr, tail, "\sim", "i"'] & & M \otimes C \\ & \Omega(M;C;C) \arrow[ur, two heads, "\widehat{\rho}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} N \arrow[rr, "\lambda"] \arrow[dr, tail, "\sim", "j"'] & & C \otimes N \\ & \Omega(C;C;N) \arrow[ur, two heads, "\widehat{\lambda}"'] & \end{tikzcd}$$

be very good two-sided cobar constructions, where $(M, \rho) \in \operatorname{Ob} \mathbf{Comod}_C$ and $(N, \lambda) \in \operatorname{Ob} {}_C\mathbf{Comod}$. If

$$M \otimes C \otimes N \text{ is fibrant} \tag{4.2}$$

and

$$(r_\rho \otimes \widehat{\lambda}, \widehat{\rho} \otimes r_\lambda) : \Omega(M;C;C) \otimes \Omega(C;C;N) \to (M \otimes C \otimes N)^{\times 2} \tag{4.3}$$

is a fibration, then

$$\operatorname{equal}\Big(\Omega(M;C;C) \otimes \Omega(C;C;N) \underset{\widehat{\rho} \otimes r_\lambda}{\overset{r_\rho \otimes \widehat{\lambda}}{\rightrightarrows}} M \otimes C \otimes N\Big),$$

which is calculated in $\mathbf{C}$, is a *model of the homotopy cotensor product* of M and N over C.

Notation 4.28. We write $M \underset{C}{\widehat{\square}} N$ for any model of the homotopy cotensor product of M and N over C. We see below that this abuse of terminology is justified, as the homotopy cotensor product construction is invariant up to weak equivalence, under mild additional conditions on M, N and C.

Theorem 4.29. *Let $(\mathbf{C}, \otimes, I)$ be a category admitting model and monoidal structures satisfying conditions (MM1) and (MM2). Let (C, Δ, ε) be a cofibrant comonoid in $\mathbf{C}$ such that $\varepsilon : C \to I$ is a fibration. Let $f : M \xrightarrow{\sim} M'$ and $g : N \xrightarrow{\sim} N'$ be weak equivalences of right and of left C-comodules with cofibrant domains, such that the pairs (M, N) and (M', N') admit models of their homotopy cotensor products over C. Then there is an induced weak equivalence*

$$f\underset{C}{\widehat{\square}}g : M\underset{C}{\widehat{\square}}N \xrightarrow{\sim} M'\underset{C}{\widehat{\square}}N',$$

for any choice of model of the respective homotopy cotensor products.

Proof. By Lemma 4.22 and Remark 4.23, for any choice of very good two-sided cobar constructions satisfying conditions (4.2) and (4.3) of Definition 4.27, there is a commutative diagram

$$\begin{tikzcd}
\Omega(M;C;C)\otimes\Omega(C;C;N) \arrow[r, two heads, "r_\rho\otimes\widehat{\lambda}"] \arrow[d, "\Omega f\otimes\Omega g", "\sim"'] & M\otimes C\otimes N \arrow[d, "f\otimes C\otimes g", "\sim"'] & \Omega(M;C;C)\otimes\Omega(C;C;N) \arrow[l, two heads, "\widehat{\rho}\otimes r_\lambda"'] \arrow[d, "\Omega f\otimes\Omega g", "\sim"'] \\
\Omega(M';C;C)\otimes\Omega(C;C;N') \arrow[r, two heads, "r_{\rho'}\otimes\widehat{\lambda}'"] & M'\otimes C\otimes N' & \Omega(M';C;C)\otimes\Omega(C;C;N) \arrow[l, two heads, "\widehat{\rho}'\otimes r_{\lambda'}"']
\end{tikzcd}$$

The three vertical arrows are weak equivalences, as they are tensor products of weak equivalences with cofibrant domain. By Corollary 4.13, since conditions (4.2) and (4.3) hold, the induced map on equalizers

$$M\underset{C}{\widehat{\square}}N \to M'\underset{C}{\widehat{\square}}N'$$

is also a weak equivalence. $\square$

It follows from Theorem 4.29 that the weak homotopy type of the homotopy cotensor product is independent of the choice of very good two-sided cobar constructions.

Corollary 4.30. *Under the hypotheses of Theorem 4.29, for any other choice of very good two-sided cobar constructions*

$$\begin{tikzcd}
M \arrow[rr, "\rho"] \arrow[dr, tail, "\sim", "i'"'] & & M\otimes C \\
& \Omega'(M;C;C) \arrow[ur, two heads, "\widehat{\rho}'"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
N \arrow[rr, "\lambda"] \arrow[dr, tail, "\sim", "j'"'] & & C\otimes N \\
& \Omega'(C;C;N) \arrow[ur, two heads, "\widehat{\lambda}'"'] &
\end{tikzcd}$$

satisfying conditions (4.2) and (4.3), there is a weak equivalence

$$M\underset{C}{\widehat{\square}}N \xrightarrow{\sim} \mathrm{equal}\Big(\Omega'(M;C;C)\otimes\Omega'(C;C;N) \underset{\widehat{\rho}'\otimes r'_\lambda}{\overset{r'_\rho\otimes\widehat{\lambda}'}{\rightrightarrows}} M\otimes C\otimes N\Big).$$

A proof similar to that of Theorem 4.29 shows that the homotopy cotensor product construction is balanced, up to weak homotopy, at least under one additional condition on $\mathbf{C}$.

Proposition 4.31. *Let $(\mathbf{C}, \otimes, I)$ be a category admitting model and monoidal structures satisfying conditions (MM1) and (MM2). Suppose furthermore that*

(MM3) $\lim : \mathbf{C}^{\mathbf{E}} \to \mathbf{C}$ *sends acyclic cofibrations with fibrant target to weak equivalences.*

Let (C, Δ, ε) be a cofibrant comonoid in $\mathbf{C}$, and let (M, ρ) and (N, λ) be a right and a left C-comodule, which are both cofibrant. Then there are weak equivalences

$$\mathrm{equal}\Big(\Omega(M;C;C) \otimes N \underset{\widehat{\rho}\otimes N}{\overset{r_\rho \otimes \lambda}{\rightrightarrows}} M \otimes C \otimes N\Big) \xrightarrow{\sim} M \underset{C}{\widehat{\square}} N$$

and

$$\mathrm{equal}\Big(M \otimes \Omega(C;C;N) \underset{\rho\otimes r_\lambda}{\overset{M \otimes \widehat{\lambda}}{\rightrightarrows}} M \otimes C \otimes N\Big) \xrightarrow{\sim} M \underset{C}{\widehat{\square}} N.$$

Proof. Consider the following diagram

$$\begin{tikzcd}
\Omega(M;C;C) \otimes N \arrow[r, "\widehat{\rho}\otimes N"] \arrow[d, "Id\otimes j", "\sim"'] & M \otimes C \otimes N \arrow[d, equal] & \Omega(M;C;C) \otimes N \arrow[l, "r_\rho \otimes \lambda"'] \arrow[d, "Id\otimes j"] \\
\Omega(M;C;C) \otimes \Omega(C;C;N) \arrow[r, "\widehat{\rho}\otimes r_\lambda"] & M \otimes C \otimes N & \Omega(M;C;C) \otimes \Omega(C;C;N), \arrow[l, "r_\rho \otimes \widehat{\lambda}"']
\end{tikzcd}$$

which is an acyclic cofibration of equalizer diagrams, with fibrant target. The additional hypothesis on $\mathbf{C}$ then implies that the induced map on equalizers is a weak equivalence. □

Definition 4.32. Let $(\mathbf{C}, \otimes, I)$ be a category admitting model and monoidal structures satisfying conditions (MM1) and (MM2). Let (C, Δ, ε) be a cofibrant comonoid in $\mathbf{C}$ such that $\varepsilon : C \to I$ is a fibration. Let (M, ρ) be a right C-comodule. An object M^{hC} *is a model of the homotopy cofixed points* of the coaction ρ if $M^{hC} = M \underset{C}{\widehat{\square}} I$ for some choice of very good two-sided cobar construction on M and of very good acyclic cobar construction on I satisfying conditions (4.2) and (4.3) or if

$$M^{hC} = \mathrm{equal}\Big(\Omega(M;C;C) \underset{\widehat{\rho}}{\overset{r_\rho}{\rightrightarrows}} M \otimes C \otimes N\Big),$$

when (MM3) is also satisfied.

Question 4.33. How does the definition of homotopy cotensor product given above relate to that in Definition 1.17? Can't say anything when $\mathbf{C}$ is arbitrary, since the comparison involves an objectwise weak equivalence in which only the target is necessarily fibrant, but the next exercise shows that the two possible definitions agree in the case $\mathbf{C} = \mathbf{Ch}$.

Exercise 4.34. (Working again with field coefficients for the sake of simplicity...) Let $\mathbf{C} = \mathbf{Ch}(\Bbbk)$, the category of chain complexes of $\Bbbk$-vector spaces, endowed with the model category structure of Exercise 4.17. Let (M, ρ) and (N, λ) be a right and a left C-comodule, where C is a coaugmented, coassociative chain coalgebra. Then

(1) the two very good two-sided cobar constructions

$$\begin{tikzcd} M \arrow[rr, "\rho"] \arrow[dr, "i"', "\sim"] & & M\otimes C \\ & M\otimes_{t_\Omega} \Omega C\otimes_{t_\Omega} C \arrow[ur, "\widehat{\rho}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} N \arrow[rr, "\rho"] \arrow[dr, "j"', "\sim"] & & C\otimes N \\ & C\otimes_{t_\Omega} \Omega C\otimes_{t_\Omega} N \arrow[ur, "\widehat{\lambda}"'] & \end{tikzcd}$$

of Exercise 4.17 satisfy condition (4.3); and

(2) the two definitions of homotopy cotensor product agree up to quasi-isomorphism in this case.

In Lecture 1, we built the two-sided cobar constructions from the (reduced) cobar construction and its universal twisting cochain. Here, we derive the reduced cobar construction from the two-sided cobar construction.

Definition 4.35. Let (C,η) be a coaugmented comonoid in $\mathbf{C}$. Let

$$I \underset{\sim}{\overset{j}{\rightarrowtail}} \Omega(C;C) \xrightarrow{\widehat{\eta}} C$$

be an acyclic cobar construction satisfying condition (4.3). A *fat reduced cobar construction* on C, denoted $\mathbf{\Omega}C$, is the equalizer of the morphisms

$$\Omega(C;C)\otimes\Omega(C;C) \xrightarrow{\widehat{\eta}\otimes\varepsilon\widehat{\eta}} C \xleftarrow{\varepsilon\widehat{\eta}\otimes\widehat{\eta}} \Omega(C;C)\otimes\Omega(C;C)\,,$$

while a *thin reduced cobar construction*, denoted ΩC, is the equalizer of the morphisms

$$\Omega(C;C) \xrightarrow{\widehat{\eta}} C \xleftarrow{\eta\varepsilon\widehat{\eta}} \Omega(C;C)\,.$$

Remark 4.36. The same sort of argument already seen above shows that if C is cofibrant and $\varepsilon : C\to I$ is a fibration, then there is a weak equivalence $\mathbf{\Omega}C\to\Omega C$.

4.4. Examples.

4.4.1. *Topological spaces.* Consider the category **Top** of topological spaces, which is monoidal with respect to the cartesian product, with unit object $\{*\}$, endowed with its Serre model structure (fibrations are Serre fibrations, while weak equivalences are maps inducing isomorphisms on homotopy groups, for any choice of basepoint). Note that additional axioms (MM1) and (MM2) clearly hold in **Top**. Furthermore, every space is a comonoid, where the comultiplication is the usual diagonal Δ. A comonoid under $\{*\}$ is therefore just a based space.

Any continuous map $f : X\to Y$ induces right and left Y-comodule structures on X, via the composites

$$X \xrightarrow{\Delta} X\times X \xrightarrow{X\times f} X\times Y$$

and

$$X \xrightarrow{\Delta} X\times X \xrightarrow{f\times X} Y\times X.$$

Inversely, every right (or left) Y-comodule structure on X gives rise to a continuous map $X \to Y$, given by the composite

$$X \xrightarrow{\rho} X \times Y \xrightarrow{pr_2} Y.$$

In other words, right (or left) Y-comodule structures on X are in bijective correspondence with continuous maps from X to Y.

The (unbased) path space PY on a space Y is clearly a very good two-sided cobar construction on Y seen as a comodule over itself, i.e., we can choose $\Omega(Y;Y;Y) = PY$, where $PY \to Y \times Y$ is given by evaluation at the endpoints of the path and $Y \to PY$ sends y to the constant path at y.

More generally, for any continuous map $f : X \to Y$, we can define $\Omega(X;Y;Y)$ and $\Omega(Y;Y;X)$ by pullback, as follows.

$$\begin{tikzcd}
\Omega(X;Y;Y) \arrow[r] \arrow[d] & \Omega(Y;Y;Y) \arrow[d, "{(ev_0,ev_1)}"] \\
X \times Y \arrow[r, "f\times Y"] & Y \times Y
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega(Y;Y;X) \arrow[r] \arrow[d] & \Omega(Y;Y;Y) \arrow[d, "{(ev_0,ev_1)}"] \\
Y \times X \arrow[r, "Y\times f"] & Y \times Y
\end{tikzcd}$$

In other words,

$$\Omega(X;Y;Y) = \{(x, \lambda) \in X \times Y^I \mid \lambda(0) = f(x)\}$$

and similarly for $\Omega(Y;Y;X)$. In particular, there are the following possible acyclic cobar constructions:

- $\Omega(Y;Y;*)$, which is the space of paths ending in the basepoint, and
- $\Omega(*;Y;Y)$, which is the space of paths starting in the basepoint.

Exercise 4.37. The usual based path space on Y is a thin reduced cobar construction on Y!

Exercise 4.38. Let $f : X \to Y$ and $f' : X' \to Y$ be continuous maps, giving rise to left and right Y-comodule structures on X and X', respectively. The homotopy cotensor product $X \underset{Y}{\widehat{\square}} X'$ is just the homotopy pullback of f and f'. In particular, the homotopy cofixed point space X^{hY} of the coaction of Y on X is just the homotopy fiber of f.

4.4.2. *Simplicial sets.* Let **sSet** denote the category of simplicial sets, which is monoidal with respect to the cartesian product, where the unit object is the constant simplicial set $*$. The additional axioms (MM1) and (MM2) hold in **sSet**, endowed with its usual model structure. Every simplicial set is a comonoid, where the comultiplication is again the usual diagonal. As in the topological case, left or right comodule structures on a simplicial K with respect to a simplicial set L are in bijective correspondence with the set of simplicial maps from K to L.

We construct our two-sided cobar constructions as two-sided twisted cartesian products of simplicial sets. As this notion is not entirely standard, we define it carefully. Recall that the definitions of twisting function and of twisted cartesian product (Definitions 3.16 and 3.18). We now introduce a two-sided version of the twisted cartesian product.

Definition 4.39. Let G be a simplicial group, and let F be a simplicial set on which there is a two-sided G-action. Given twisting functions $\tau : K_\bullet \to G_{\bullet-1}$ and $\upsilon : L_\bullet \to G_{\bullet-1}$, let $K \times_\tau F \times_\upsilon L$ be the simplicial set with underlying graded set $K \times F \times L$, such that all face and degeneracy maps act componentwise, except the 0-face, where

$$\partial_0(x, y, z) := (\partial_0 x, \tau(x) \cdot \partial_0 y \cdot \upsilon(z)^{-1}, \partial_0 z).$$

Exercise 4.40. The two-sided twisted cartesian product as defined above is indeed a simplicial set.

Proposition 4.41. *Let $f : K \to L$ be a simplicial map, where L is a reduced simplicial set. Let*

$$\Omega(K; L; L) = K \underset{\tau_L f}{\times} GL \underset{\tau_L}{\times} L \quad \text{and} \quad \Omega(L; L; K) = L \underset{\tau_L}{\times} GL \underset{\tau_L f}{\times} K,$$

where GL acts on itself on the right and on the left by multiplication Let $j : K \to \Omega(K; L; L)$ and $j : K \to \Omega(L; L; K)$ be the two natural inclusions, and let $\widehat{\rho} : \Omega(K; L; L) \to K \times L$ and $\widehat{\lambda} : \Omega(L; L; K) \to L \times K$ be the two natural projections. Then

$$K \xrightarrow{j} \Omega(K; L; L) \xrightarrow{\widehat{\rho}} K \times L$$

and

$$K \xrightarrow{j} \Omega(L; L; K) \xrightarrow{\widehat{\lambda}} L \times K$$

are natural, very good two-sided cobar constructions, for K seen as a right, respectively left, L-comodule via f.

Corollary 4.42. *The Kan loop group GL on a reduced simplicial set L is a thin reduced cobar construction on L.*

As in the topological case, homotopy cotensor product corresponds to homotopy pullback, so that the homotopy cofixed point set of a coaction correponds to homotopy fiber.

4.5. **Cobar constructions via totalization.** (Somewhat more technical, so just a slightly informal sketch...)

Definition 4.43. A category $\mathbf{C}$ is *simplicial* if

- for all $A, B \in \operatorname{Ob} \mathbf{C}$, there is a simplicial set $\operatorname{Map}(A, B)$ such that $\operatorname{Map}(A, B) \cong \mathbf{C}(A, B)$;
- for all $A, B, C \in \operatorname{Ob} \mathbf{C}$, there is a *composition rule*

  $$c : \operatorname{Map}(B, C) \times \operatorname{Map}(A, B) \to \operatorname{Map}(A, C);$$

  and
- for all A, there is a *unit map* $i_A : \Delta[0] \to \operatorname{Map}(A, A)$ such that the family of composition rules is associative and unital with respect to the unit maps.

A model category $\mathbf{C}$ that is also simplicial is a *simplicial model category* if the following two additional axioms are satisfied.

(M6) For all $A, B \in \operatorname{Ob} \mathbf{C}$ and for all simplicial sets K, there exist $A \otimes K, B^K \in \operatorname{Ob} \mathbf{C}$ together with natural isomorphisms of simplicial sets

$$\operatorname{Map}(A \otimes K, B) \cong \operatorname{Map}\big(K, \operatorname{Map}(A, B)\big) \cong \operatorname{Map}(A, B^K).$$

(M7) For all $i : A \rightarrowtail B$ and $p : X \twoheadrightarrow Y$, the induced map of simplicial sets

$$\mathrm{Map}(B,X) \to \mathrm{Map}(A,X) \underset{\mathrm{Map}(A,Y)}{\times} \mathrm{Map}(B,Y)$$

is a Kan fibration, which is acyclic if either i or p is acyclic.

Example 4.44. The obvious example: $\mathbf{C} = \mathbf{sSet}$, with $\mathrm{Map}(A,B)_n = \mathbf{sSet}(A \times \Delta[n], B)$, the usual simplicial mapping space, where $\Delta[n]$ is the standard n-simplex. Furthermore, $A \otimes K = A \times K$ and $B^K = \mathrm{Map}(B,K)$.

Example 4.45. A slightly more sophisticated example: $\mathbf{C} = \mathbf{Top}$, with $\mathrm{Map}(A,B)_n = \mathbf{Top}(A \times \Delta^n, B)$, where Δ^n is the topological n-simplex. We then set $A \otimes K = A \times |K|$, while $B^K = B^{|K|}$.

Definition 4.46. Let $\mathbf{C}$ be a simplicial model category. Let $X^\bullet : \mathbf{\Delta} \to \mathbf{C}$ be a cosimplicial object in $\mathbf{C}$. The *totalization* of $X^\bullet$ is

$$\mathrm{Tot}\, X^\bullet = \mathrm{equal}\Big(\prod_{n\in\mathbb{N}} (X^n)^{\Delta[n]} \rightrightarrows \prod_{f\in\mathbf{\Delta}(n,k)} (X^k)^{\Delta[n]} \Big),$$

which is an object in $\mathbf{C}$.

Remark 4.47. This can be generalized to model categories endowed with a "Reedy framing."

Notation 4.48. Given $A \in \mathrm{Ob}\, C$, let $\underline{A}^\bullet$ denote the constant cosimplicial object on A, i.e., $A^n = A$ for all n and all faces and degeneracies are identities.

Remark 4.49. It is relatively easy to see that in a simplicial model category, there are weak equivalences

$$\mathrm{Tot}\, \underline{A}^\bullet \xrightarrow{\simeq} \mathrm{hocolim}\, \underline{A}^\bullet \xrightarrow{\simeq} \lim \underline{A}^\bullet \cong A.$$

Let $\pi : \mathrm{Tot}\, \underline{A}^\bullet \to A$ denote this composite. In fact, π admits a section $s : A \to \mathrm{Tot}\, \underline{A}^\bullet$, induced by the morphisms $A^{\Delta[0]} \to A^{\Delta[n]}$, which, in turn, are induced by the simplicial maps $\Delta[n] \to \Delta[0]$. The section s must also be a weak equivalence.

Definition 4.50. Let C be a comonoid in a monoidal category $\mathbf{C}$. Let (M, ρ) be a right C-comodule. The *two-sided cosimplicial cobar construction* on M is a cosimplicial object $\Omega^\bullet(M; C; C)$ in $\mathbf{C}$ such that

- $\Omega^n(M; C; C) = M \otimes C^{\otimes n+1}$, for all n;
- $d^0 = \rho \otimes C^{\otimes n+1} : \Omega^n(M; C; C) \to \Omega^{n+1}(M; C; C)$;
- $d^i = M \otimes C^{\otimes i-1} \otimes \Delta \otimes C^{\otimes n+1-i} : \Omega^n(M; C; C) \to \Omega^{n+1}(M; C; C)$ for all $1 \leq i \leq n+1$; and
- $s^j = M \otimes C^{\otimes j} \otimes \varepsilon \otimes C^{\otimes n-j} : \Omega^n(M; C; C) \to \Omega^{n-1}(M; C; C)$ for all $0 \leq j \leq n$.

Exercise 4.51. Let C be a comonoid in a monoidal category $\mathbf{C}$. Let (M, ρ) be a right C-comodule. Define $j^\bullet : M^\bullet \to \Omega^\bullet(M; C; C)$ by

$$j^n = \rho^{(n+1)} : M \to M \otimes C^{\otimes n+1},$$

obtained by applying the coaction iteratively, $n+1$ times. Define $\widehat{\rho}^\bullet : \Omega(M; C; C) \to (M \otimes C)^\bullet$ by

$$\widehat{\rho}^n = M \otimes \varepsilon^{\otimes n} \otimes C : M \otimes C^{\otimes n+1} \to M \otimes C.$$

Then $j^\bullet$ and $\widehat{\rho}^\bullet$ are both cosimplicial maps, i.e., they commute with cofaces and codegeneracies.

Proposition 4.52. *Using the notation of the exercise above, let* $j = \operatorname{Tot} j^\bullet \circ s$*, and let* $\widehat{\rho} = \pi \operatorname{Tot} \widehat{\rho}^\bullet$*. Then*

$$M \xrightarrow[\simeq]{j} \operatorname{Tot}\Omega^\bullet(M;C;C) \xrightarrow{\widehat{\rho}} M \otimes C$$

is a two-sided cobar construction on M*.*

Proof. The usual methods show that $j^\bullet$ is a cosimplicial homotopy equivalence and therefore $\operatorname{Tot} j^\bullet$ is a weak equivalence, by a theorem of Bousfield. □

Remark 4.53. Probably need more precise knowledge of the model category $\mathbf{C}$ and of the morphisms ρ and ε in order to determine when the two-sided cobar construction above is good or very good or respects coactions.

4.6. **Naturality and Eilenberg-Moore theorems.**

Question 4.54. When do functors between monoidal model categories preserve homotopy cotensor products, up to weak equivalence?

Inspiration for this question comes from...

Theorem 4.55 (Eilenberg-Moore)**.** *For any (Serre) fibration* $E \to B$ *such that* B *is connected and simply connected, any continuous map* $f : X \to B$ *of simply connected spaces and any commutative ring* R*, there is an* R*-linear isomorphism*

$$\mathrm{H}_*(E \underset{B}{\times} X; R) \cong \operatorname{Cotor}^{C_*(B;R)}\big(C_*(E;R), C_*(X;R)\big). \tag{4.4}$$

It's an easy exercise to prove the following lemma.

Lemma 4.56. *Let* $(\mathbf{C}, \otimes, I)$ *and* $(\mathbf{D}, \otimes, J)$ *be model categories endowed with monoidal structure and satisfying axioms (MM1) and (MM2). Let* $F : \mathbf{C} \to \mathbf{D}$ *be a functor such that*

(EM1) F *preserves fibrant objects, as well as fibrations and acyclic cofibrations;*
(EM2) F *is comonoidal, i.e., there is an appropriately coassociative and counital natural transformation*

$$\delta : F(- \otimes -) \to F(-) \otimes F(-);$$

and

(EM3) *every component of* δ *is a fibration in* $\mathbf{D}$*, i.e.,*

$$\delta_{A,B} : F(A \otimes B) \twoheadrightarrow F(A) \otimes F(B)$$

for all $A, B \in \operatorname{Ob}\mathbf{C}$*.*

Then F *preserves comonoids and very good two-sided cobar constructions.*

To prove an Eilenberg-Moore-type theorem, we need further conditions on our functor F.

Definition 4.57. Let $(\mathbf{C}, \otimes, I)$ and $(\mathbf{D}, \otimes, J)$ be model categories endowed with monoidal structure and satisfying axioms (MM1) and (MM2). A functor $F : \mathbf{C} \to \mathbf{D}$ is an *Eilenberg-Moore functor* if it satisfies conditions (EM1) and (EM2) of Lemma 4.56, as well as

(EM3′) every component of δ is an *acyclic* fibration in $\mathbf{D}$, i.e.,

$$\delta_{A,B} : F(A \otimes B) \overset{\sim}{\twoheadrightarrow} F(A) \otimes F(B)$$

for all $A, B \in \operatorname{Ob}\mathbf{C}$;

(EM4) the natural transformation δ admits a section

$$\sigma : F(-) \otimes F(-) \to F(- \otimes -),$$

i.e., σ is a natural transformation such that the composite

$$F(A) \otimes F(B) \xrightarrow{\sigma_{A,B}} F(A \otimes B) \xrightarrow{\delta_{A,B}} F(A) \otimes F(B)$$

is the identity morphism for all $A, B \in \operatorname{Ob} \mathbf{C}$;

(EM5) the obvious natural transformation

$$\pi : F(- \times -) \to F(-) \times F(-)$$

is a fibration in every component, i.e.,

$$\pi_{A,B} = \big(F(pr_1), F(pr_2)\big) : F(A \times B) \twoheadrightarrow F(A) \times F(B)$$

for all $A, B \in \operatorname{Ob} \mathbf{C}$; and

(EM6) for every diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathbf{C}$ where $(f, g) : A \to B \times B$ is a fibration and B is fibrant (i.e., $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is fibrant in $\mathbf{C}^{\mathbf{E}}$) the induced map

$$F\big(\operatorname{equal}(A \underset{g}{\overset{f}{\rightrightarrows}} B)\big) \to \operatorname{equal}\big(F(A) \underset{F(g)}{\overset{F(f)}{\rightrightarrows}} F(B)\big)$$

is a weak equivalence in $\mathbf{D}$.

Remark 4.58. There are certainly other and probably better choices of conditions to impose on a functor to insure that an Eilenberg-Moore-type theorem holds, but this choice works and seems to me to be fairly natural and reasonable.

Theorem 4.59. *Let $(\mathbf{C}, \otimes, I)$ and $(\mathbf{D}, \otimes, J)$ be model categories endowed with monoidal structure and satisfying axioms (MM1) and (MM2). Let $F : \mathbf{C} \to \mathbf{D}$ be an Eilenberg-Moore functor. Let C be a comonoid and M and N a right and a left C-comodule in $\mathbf{C}$, which are all fibrant and cofibrant as objects in $\mathbf{C}$. Then there is a weak equivalence in $\mathbf{D}$*

$$F(M \underset{C}{\widehat{\square}} N) \xrightarrow{\sim} F(M) \underset{F(C)}{\widehat{\square}} F(N).$$

Proof. Let

$$\begin{tikzcd} M \arrow[rr, "\rho"] \arrow[dr, tail, "\sim", "i"'] & & M \otimes C \\ & \Omega(M; C; C) \arrow[ur, two heads, "\widehat{\rho}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} N \arrow[rr, "\lambda"] \arrow[dr, tail, "\sim", "j"'] & & C \otimes N \\ & \Omega(C; C; N) \arrow[ur, two heads, "\widehat{\lambda}"'] & \end{tikzcd}$$

be very good two-sided cobar constructions on M and N satisfying conditions (4.2) and (4.3), inducing very good two-sided cobar constructions on $F(M)$ and $F(N)$:

$$\begin{tikzcd} F(M) \arrow[rr, "\delta_{M,C}F(\rho)"] \arrow[dr, tail, "\sim", "F(i)"'] & & F(M)\otimes F(C) \\ & F\big(\Omega(M;C;C)\big) \arrow[ur, two heads, "\delta_{M,C}F(\widehat\rho)"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} F(N) \arrow[rr, "\delta_{C,N}F(\lambda)"] \arrow[dr, tail, "\sim", "F(j)"'] & & F(C)\otimes F(N)\ . \\ & F\big(\Omega(C;C;N)\big) \arrow[ur, two heads, "\delta_{C,N}F(\widehat\lambda)"'] & \end{tikzcd}$$

By condition (EM6), the natural, induced map

$$F(M\underset{C}{\widehat\square}N)=F\big(\operatorname{equal}(\Omega(M;C;C)\otimes\Omega(C;C;N)\underset{r_\rho\otimes\widehat\lambda}{\overset{\widehat\rho\otimes r_\lambda}{\rightrightarrows}}M\otimes C\otimes N)\big)$$

$$\Big\downarrow$$

$$\operatorname{equal}\big(F(\Omega(M;C;C)\otimes\Omega(C;C;N))\underset{F(r_\rho\otimes\widehat\lambda)}{\overset{F(\widehat\rho\otimes r_\lambda)}{\rightrightarrows}}F(M\otimes C\otimes N)\big)$$

is a weak equivalence. Furthermore, (EM1) and (EM5) together imply that the composite

$$\begin{tikzcd} F(\Omega(M;C;C)\otimes\Omega(C;C;N)) \arrow[r, "F(\widehat\rho\otimes r_\lambda,r_\rho\otimes\widehat\lambda)"] \arrow[dr, "\big(F(\widehat\rho\otimes r_\lambda),F(r_\rho\otimes\widehat\lambda)\big)"'] & F\big((M\otimes C\otimes N)^{\times 2}\big) \arrow[d, "\pi"] \\ & F(M\otimes C\otimes N)^{\times 2} \end{tikzcd}$$

is a fibration, since the pair of two-sided cobar constructions on M and N satisfies condition (4.3). Condition (EM1) also implies that $F(M\otimes C\otimes N)$ is fibrant, and therefore that the equalizer diagram

$$F(\Omega(M;C;C)\otimes\Omega(C;C;N))\underset{F(r_\rho\otimes\widehat\lambda)}{\overset{F(\widehat\rho\otimes r_\lambda)}{\rightrightarrows}}F(M\otimes C\otimes N)\tag{4.5}$$

is fibrant.

Condition (EM4) now implies that $F(M)\otimes F(C)\otimes F(N)$ is fibrant, since it is a retract of a fibrant object, and that

$$\big(\delta F(\widehat\rho)\otimes F(r_\lambda),F(r_\rho)\otimes\delta F(\widehat\lambda)\big):F(\Omega(M;C;C))\otimes F(\Omega(C;C;N))\to(M\otimes C\otimes N)^{\times 2}$$

is a fibration, since δ is a fibration in every component and $\big(F(\widehat\rho)\otimes F(r_\lambda),F(r_\rho)\otimes F(\widehat\lambda)\big)$ is a retract of $\big(F(\widehat\rho\otimes r_\lambda),F(r_\rho\otimes\widehat\lambda)\big)$. The equalizer diagram

$$F(\Omega(M;C;C))\otimes F(\Omega(C;C;N))\underset{F(r_\rho)\otimes\delta F(\widehat\lambda)}{\overset{\delta F(\widehat\rho)\otimes F(r_\lambda)}{\rightrightarrows}}F(M)\otimes F(C)\otimes F(N)\tag{4.6}$$

is therefore fibrant.

To conclude, observe that the natural transformation δ gives rise to an objectwise weak equivalence from fibrant diagram (4.5) to fibrant diagram (4.6) and therefore induces a weak equivalence on their limits. $\square$

Example 4.60. (Working with the model structure on $\mathbf{Ch}(\Bbbk)$ of Exercise 4.17) The normalized chains functor $C_* : \mathbf{sSet} \to \mathbf{Ch}(\Bbbk)$ is an Eilenberg-Moore functor. It clearly satisfies (EM1) and (EM5). The natural Alexander-Whitney transformation

$$f_{K,L} : C_*(K \times L) \to C_*(K) \otimes C_*(L)$$

gives the comonoidal structure of $C_*(-)$ and is an acyclic fibration in each component, so that (EM2) and (EM3′) hold. The section of f required for (EM4) is just the Eilenberg-Zilber natural equivalence, ∇. A relatively simple argument shows that if $K \underset{g}{\overset{f}{\rightrightarrows}} L$ is a fibrant equalizer diagram in $\mathbf{sSet}$, then the natural, induced map

$$C_*\big(\operatorname{equal}(K \underset{g}{\overset{f}{\rightrightarrows}} L)\big) \xrightarrow{\simeq} \operatorname{equal}\big(C_*K \underset{C_*g}{\overset{C_*f}{\rightrightarrows}} C_*L\big)$$

is a quasi-isomorphism, i.e., (EM6) holds as well. In fact, it probably suffices to suppose that f is a fibration in order to conclude that the induced map is a quasi-isomorphism.

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