Deligne's Hochschild Cohomology Conjecture

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Outline

Definition

[Hochschild, 1945]

Let $\Bbbk$ be any commutative ring, and let A be an associative $\Bbbk$-algebra.

The Hochschild cochain complex of A is

$$C^*(A,A) = \left(C^0(A,A) \xrightarrow{d^0} C^1(A,A) \xrightarrow{d^1} C^2(A,A) \xrightarrow{d^2} \cdots\right)$$

where

- $C^n(A,A) = \hom(A^{\otimes n}, A)$,
- d^n defined in terms of the multiplication on A.

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and the Hochschild cohomology of A is

$$H^*(A,A) = H^*\big(C^*(A,A)\big).$$

Why interesting?

$H^*(A, A)$ classifies infinitesimal deformations of the multiplicative structure of A.

Structure

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- a graded $\mathbb{k}$-module,
- endowed with a graded commutative multiplication:

$$H^*(A, A) \otimes H^*(A, A) \to H^*(A, A) : \alpha \otimes \beta \mapsto \alpha \cdot \beta$$

$$\alpha \in H^m(A, A), \beta \in H^n(A, A) \implies \alpha \cdot \beta = (-1)^{mn} \beta \cdot \alpha \in H^{m+n}(A, A).$$

Structure

$H^*(A, A)$ is

- a graded $\mathbb{k}$-module,
- endowed with a graded commutative multiplication,
- and a Lie bracket of degree -1:

$$H^*(A, A) \otimes H^*(A, A) \to H^*(A, A) : \alpha \otimes \beta \mapsto [\alpha, \beta]$$

$$\alpha \in H^m(A, A), \beta \in H^n(A, A).$$

$$\Longrightarrow [\alpha, \beta] \in H^{m+n-1}(A, A).$$

Structure

$H^*(A, A)$ is

- a graded $\Bbbk$-module,
- endowed with a graded commutative multiplication,
- and a Lie bracket of degree -1 satisfying a graded Jacobi identity, graded anticommutativity and such that

$$[\alpha, \beta \cdot \gamma] = [\alpha, \beta] \cdot \gamma + (-1)^{m(n+1)} \beta \cdot [\alpha, \gamma]$$

for all $\alpha \in H^m(A, A), \beta \in H^n(A, A), \gamma \in H^*(A, A)$.

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Question

How to encode this complicated algebraic structure as compactly as possible?

Definition

An operad $\mathcal{O}$ consists of

- a sequence of $\mathbb{k}$-modules

$$\mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), \ldots;$$

- a collection of $\mathbb{k}$-linear maps

$$\mathcal{O}(k) \otimes \big(\mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k)\big) \longrightarrow \mathcal{O}\Big(\sum_{i=1}^{k} n_i\Big)$$

satisfying reasonable associativity and unitality conditions.

The endomorphism operad

Let X be a $\Bbbk$-module. The endomorphism operad $\mathcal{E}_X$ is given by

- $\mathcal{E}_X(n) = \hom(X^{\otimes n}, X)$,
- if $n = \sum_{i=1}^{k} n_i$, then

$$\mathcal{E}_X(k) \otimes \big(\mathcal{E}_X(n_1) \otimes \cdots \otimes \mathcal{E}_X(n_k)\big) \to \mathcal{E}_X(n)$$

sends

$$f \otimes (g_1 \otimes \cdots \otimes g_k)$$

to

$$f \circ (g_1 \otimes \cdots \otimes g_k) : X^{\otimes n} \to X.$$

Operad maps

Let $\mathcal{O}$ and $\mathcal{P}$ be operads.

An operad map $\varphi : \mathcal{O} \to \mathcal{P}$ consists of $\Bbbk$-linear maps

$$\varphi_n : \mathcal{O}(n) \to \mathcal{P}(n), \quad n \geq 0$$

such that

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes \big(\mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k)\big) & \longrightarrow & \mathcal{O}(n) \\
\Big\downarrow {\scriptstyle \varphi_k \otimes (\varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k})} & & \Big\downarrow {\scriptstyle \varphi_n} \\
\mathcal{P}(k) \otimes \big(\mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k)\big) & \longrightarrow & \mathcal{P}(n)
\end{array}$$

commutes for all k, n_1,... n_k.

Algebras over an operad

Let $\mathcal{O}$ be an operad.

An $\mathcal{O}$-algebra consists of a $\Bbbk$-module X and an operad map

$$\mathcal{O} \to \mathcal{E}_X.$$

Thus:

$\mathcal{O}$-algebra structure on X = representation of $\mathcal{O}$ on X.

Other categories

In the discussion above of operads and their algebras, we could replace $\mathbb{k}$-modules and $\mathbb{k}$-linear maps everywhere by

- sets and set maps,
- topological spaces and continuous maps,
- chain complexes and chain maps.

The Gerstenhaber operad

There is an operad $\mathcal{G}$ such that

$$\mathcal{G}\text{-algebras} = \text{Gerstenhaber algebras.}$$

In particular, for every associative $\Bbbk$-algebra A, there is an operad map

$$\varphi : \mathcal{G} \to \mathcal{E}_{H^*(A,A)}$$

parametrizing the natural Gerstenhaber algebra structure on Hochschild cohomology.

The little discs operad

Let

$\mathcal{D}(n) = \{$configurations of n discs within the unit disc in $\mathbb{R}^2\}$

(topologized appropriately) and define

$$\mathcal{D}(k) \times \big(\mathcal{D}(n_1) \times \cdots \times \mathcal{D}(n_k)\big) \to \mathcal{D}\Big(\sum_{i=1}^{k} n_i\Big)$$

by embedding configurations.

The chain little discs operad is

$$\mathcal{S} = S_*(\mathcal{D}; \mathbb{k}).$$

Little discs and Gerstenhaber

- $\mathcal{D}$ detects double loop spaces:

$$X \text{ a } \mathcal{D}\text{-algebra} \iff \exists Y \text{ such that } X \sim \Omega^2 Y.$$

- [Cohen, 1976] $H_*\mathcal{D} = H_*\mathcal{S} = \mathcal{G}$, whence

$$C \text{ an } \mathcal{S}\text{-algebra} \Longrightarrow H_*C \text{ a } \mathcal{G}\text{-algebra}.$$

Question

For which chain complexes C can the second implication be reversed? Can it be reversed for $C = C^*(A, A)$?

Deligne's letter

[1993]

"I would like the complex computing Hochschild cohomology to be an algebra over the operad $\mathcal{S}$ (or a suitable version of it)."

Conjecture

For any associative $\Bbbk$-algebra A, the Hochschild complex $C^*(A, A)$ is an $\mathcal{S}'$-algebra for some operad $\mathcal{S}'$ that is "equivalent" to the chain little discs operad $\mathcal{S}$.

Gerstenhaber-Voronov, 1994

Constructed an operad $\mathcal{H}$ that parametrized the explicit "up-to-homotopy Gerstenhaber algebra"-structure of $C^*(A, A)$ and a representation

$$\mathcal{H} \to \mathcal{E}_{C^*(A,A)}.$$

Left open the question of the relationship between $\mathcal{S}$ and $\mathcal{H}$.

Tamarkin, 1998

Constructed an operad $\mathcal{G}_\infty$ (a sort of minimal resolution of $\mathcal{G}$) and operad maps

$$\mathcal{S} \xleftarrow{\sim} \mathcal{G}_\infty \to \mathcal{E}_{C^*(A,A)}.$$

The first real proof of Deligne's conjecture, though left open the question of the existence of a representation of $\mathcal{S}$ directly on $C^*(A, A)$.

Voronov, 1999

Constructed an operad $\widetilde{\mathcal{S}}$ (a "geometric resolution" of $\mathcal{S}$) and operad maps

$$\mathcal{S} \xleftarrow{\sim} \widetilde{\mathcal{S}} \to \mathcal{H} \to \mathcal{E}_{C^*(A,A)}.$$

McClure-Smith, 1999

Constructed a “cellular” topological operad $\mathcal{C}$ equivalent to $\mathcal{D}$ and an operad map

$$S_*(\mathcal{C};\mathbb{k}) \xrightarrow{\sim} \mathcal{H},$$

whence

$$\mathcal{S} \xleftarrow{\sim} S_*(\mathcal{C};\mathbb{k}) \xrightarrow{\sim} \mathcal{H} \to \mathcal{E}_{C^*(A,A)}$$

and

$$H_*\mathcal{H} \cong H_*\mathcal{C} \cong H_*\mathcal{D} \cong \mathcal{G},$$

answering a question left open by Gerstenhaber and Voronov.

The first “geometric” proof of Deligne’s conjecture: the operad $S_*(\mathcal{C};\mathbb{k})$ comes from topology and acts directly on $C^*(A, A)$.

Kontsevich-Soibelman, 2000

Constructed an operad $\mathcal{P}$ (a "tree resolution" of $\mathcal{S}$) and operad maps

$$\mathcal{S} \xleftarrow{\sim} \mathcal{P} \to \mathcal{E}_{C^*(A,A)}.$$

McClure-Smith, 2001 and 2002

Refined and simplified considerably their original proof, establishing a generalization of the “geometric” Deligne conjecture.

Berger-Fresse, 2001

Constructed an operad $\mathcal{F}$ of simplicial sets such that

$$|\mathcal{F}| \sim \mathcal{D}$$

and an operad map

$$C_*\mathcal{F} \to \mathcal{E}_{C^*(A,A)},$$

where C_* denotes the normalized chains functor.

Another "geometric" proof.

Other recent proofs

In [Kaufmann-Schwell, 2007] and [Batanin-Berger, 2009], nice, "small" operads were constructed that are

- equivalent to $\mathcal{S}$, and
- act directly on $C^*(A, A)$.

Various generalizations

- [Hu-Kriz-Voronov, 2003] Proved that if A is an $\mathcal{E}_n$-algebra, then $C^*(A, A)$ is an $\mathcal{E}_{n+1}$-algebra.
- [Costello, 2004] Obtained a generalized version of the Deligne conjecture as corollary of an important theorem about topological conformal field theories.
- [Kontsevich-Soibelman, 2006] Proved that the pair $\big(C^*(A, A), C_*(A, A)\big)$ is an algebra over the chains on a certain colored operad.
- [Vallette, 2006] Generalized Deligne's conjecture to algebras over any finitely generated, binary, nonsymmetric Koszul operad.

What?

A concrete version of Deligne's conjecture...

Theorem (H.-Scott, 2010)

If A is an associative $\Bbbk$-algebra, let $\operatorname{Def}(A)$ be the space of homotopy deformations of the multiplication on A. Then

$$\pi_*\Omega^2 \operatorname{Def}(A) \cong H^*(A, A).$$

Connection with Deligne conjecture...

- $\Omega^2 \operatorname{Def}(A)$ is a $\mathcal{D}$-algebra, since $\mathcal{D}$ detects double loop spaces.
- Back to the roots: $H^*(A, A)$ classifies infinitesimal deformations of the multiplication on A.

Why?

Known proofs

- construct an operad $\mathcal{X}$;
- show that it is equivalent to $\mathcal{S}$;and
- show that it acts on $C^*(A, A)$.

Our proof is

- purely homotopy-theoretic;
- makes explicit the link between deformation theory of algebras and Hochschild cohomology;
- lifts the Gerstenhaber algebra structure on $H^*(A, A)$ all the way up to the level of topology.

How?

The concrete Deligne conjecture is proved by two applications of...

Theorem (Dwyer-H., 2010)

If $(\mathbf{M}, \otimes, I)$ *is a "nice enough" monoidal model category, then for any map of monoids* $\varphi : A \to B$*, there is a fiber sequence*

$$\Omega \operatorname{Map}_{\mathbf{Mon}}(A, B) \to \operatorname{Map}_{\mathbf{Bimod}}(A, B) \to \operatorname{Map}_{\mathbf{M}}(I, B).$$