Twisting Structures

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- Emmanuel Dror Farjoun (application to homotopy (co)normality)

Outline

The framework

Let $\mathbb{k}$ be a commutative ring.

- $\mathbf{Coalg}_{\mathbb{k}}$ = the category of 1-connected, coaugmented chain coalgebras over $\mathbb{k}$.
- $\mathbf{Alg}_{\mathbb{k}}$ = the category of connected, augmented chain algebras over $\mathbb{k}$.

The adjunction

$$\Omega : \mathbf{Coalg}_{\Bbbk} \rightleftarrows \mathbf{Alg}_{\Bbbk} : \mathscr{B}$$

is an adjoint pair with unit

$$\eta : \mathrm{Id} \to \mathscr{B}\Omega$$

and counit

$$\varepsilon : \Omega\mathscr{B} \to \mathrm{Id},$$

which are objectwise quasi-isomorphisms.

Twisting cochains

Given

- $(C, d_C) \in \mathbf{Coalg}_{\Bbbk}$ with comultiplication Δ,
- $(A, d_A) \in \mathbf{Alg}_{\Bbbk}$ with multiplication μ,

a linear map of degree -1

$$t : C \to A$$

is a twisting cochain if

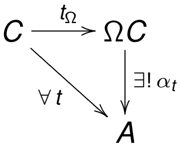
$$d_A t + t d_C = \mu(t \otimes t)\Delta.$$

The universal twisting cochain

The universal twisting cochain on $C \in \mathbf{Coalg}_{\mathbb{k}}$:

$$t_\Omega : C \to \Omega C.$$

All twisting cochains $t : C \to A$ factor uniquely through t_Ω.



The couniversal twisting cochain

The couniversal twisting cochain to $A \in \mathbf{Alg}_{\Bbbk}$:

$$t_{\mathscr{B}} : \mathscr{B}A \to A.$$

All twisting cochains $t : C \to A$ factor uniquely through $t_{\mathscr{B}}$.

$$\begin{array}{ccc} C & & \\ {\scriptstyle \exists!\,\beta_t}\downarrow & \searrow^{\forall\, t} & \\ \mathscr{B}A & \xrightarrow{t_{\mathscr{B}}} & A \end{array}$$

Twisted tensor products

Given

- $t : C \to A$, a twisting cochain;
- M a right C-comodule, with C-coaction $\rho : M \to M \otimes C$;
- N a left A-module, with A-action $\lambda : A \otimes N \to N$.

The twisted tensor product of M and N over t is a chain complex

$$M \otimes_t N = (M \otimes N, D_t),$$

where

$$D_t = d_M \otimes N + M \otimes d_N - (M \otimes \lambda)(M \otimes t \otimes N)(\rho \otimes N).$$

The universal bundles

- $t_\Omega : C \to \Omega C \implies \mathscr{P}C = C \otimes_{t_\Omega} \Omega C \simeq \Bbbk.$
- $t_\mathscr{B} : \mathscr{B}A \to A \implies \mathscr{E}A = \mathscr{B}A \otimes_{t_\mathscr{B}} A \simeq \Bbbk.$

If $t : C \to A$ is a twisting cochain, M is a right C-comodule and N is a left A-module, then

$$\begin{array}{c} M\square_C \mathscr{P}C \otimes_{\Omega C} N \\ \downarrow \cong \\ M \otimes_t N \\ \uparrow \cong \\ M\square_{\mathscr{B}A}\mathscr{E}A \otimes_A N. \end{array}$$

Every twisted tensor product can be constructed from the universal bundles, by equalizing coactions and coequalizing actions.

The framework

- $\mathbf{sSet}_0 =$ the category of reduced simplicial sets.
- $\mathbf{sGr} =$ the category of simplicial groups.

The adjunction

$$\mathscr{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{\mathscr{W}}$$

is an adjoint pair with unit

$$\eta : \mathrm{Id} \to \overline{\mathscr{W}}\mathscr{G}$$

and counit

$$\varepsilon : \mathscr{G}\overline{\mathscr{W}} \to \mathrm{Id},$$

which are objectwise weak equivalences.

Twisting functions

Given

- X, a simplicial set, and
- G a simplicial group,

a degree -1 map of graded sets

$$\tau : X \to G$$

is a twisting function if

$$\begin{aligned} d_0\tau(x) &= \big(\tau(d_0x)\big)^{-1}\tau(d_1x) \\ d_i\tau(x) &= \tau(d_{i+1}x), \quad i > 0 \\ s_i\tau(x) &= \tau(s_{i+1}x), \quad i \geq 0 \\ \tau(s_0x) &= e \end{aligned}$$

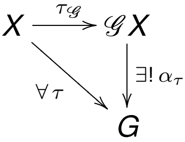
for all $x \in X$.

The universal twisting function

The universal twisting function on $X \in \mathbf{sSet}_0$:

$$\tau_{\mathscr{G}} : X \to \mathscr{G}X.$$

All twisting functions $\tau : X \to G$ factor uniquely through $\tau_{\mathscr{G}}$.

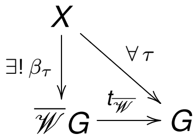


The couniversal twisting function

The couniversal twisting function to $G \in \mathbf{sGr}$:

$$\tau_{\overline{\mathscr{W}}} : \overline{\mathscr{W}}G \to G.$$

All twisting functions $\tau : X \to G$ factor uniquely through $\tau_{\overline{\mathscr{W}}}$.



Twisted cartesian products

Given

- $\tau : X \to G$, a twisting function,
- a left action of G on a simplicial set Z,
- a simplicial map $f : Y \to X$ ($\Leftrightarrow$ a right coaction of (X, Δ) on Y)

Twisted cartesian products

Given

- $\tau : X \to G$, a twisting function,
- a left action of G on a simplicial set Z,
- a simplicial map $f : Y \to X$.

The twisted cartesian product of Y and Z, denoted $Y \times_\tau Z$, is a simplicial set such that

$$(Y \times_\tau Z)_n = Y_n \times Z_n,$$

with faces and degeneracies given by

$$\begin{aligned} d_0(y,z) &= \big(d_0 y, \tau\big(f(x)\big) \cdot d_0 z\big) \\ d_i(y,z) &= (d_i y, d_i z), \quad i > 0 \\ s_i(y,z) &= (s_i y, s_i z), \quad i \geq 0. \end{aligned}$$

The universal bundles

- $\tau_{\mathscr{G}} : X \to \mathscr{G}X \implies \mathscr{P}X = X \times_{\tau_{\mathscr{G}}} \mathscr{G}X \simeq *$.
- $\tau_{\overline{\mathscr{W}}} : \overline{\mathscr{W}}G \to G \implies \mathscr{E}G = \overline{\mathscr{W}}G \times_{\tau_{\overline{\mathscr{W}}}} G \simeq *$.

If $\tau : X \to G$ is a twisting cochain, $Y \to X$ is a simplicial map and Z is a left G-module, then

$$\begin{array}{c} Y \underset{X}{\times} \mathscr{P}X \times_{\mathscr{G}X} Z \\ \downarrow \cong \\ Y \times_\tau Z \\ \uparrow \cong \\ Y \underset{\overline{\mathscr{W}}G}{\times} \mathscr{E}G \times_G Z. \end{array}$$

Every twisted cartesian product can be constructed from the universal bundles by pullbacks and by coequalizing group actions.

The normalized chains functor

Theorem (Aguiar-Mahajan)

The normalized chains functor

$$C_* : \mathbf{sSet}_0 \to \mathbf{Ch}_{\Bbbk}$$

is a *normal bilax monoidal functor:*

- *lax monoidal, via the Eilenberg-Zilber map*

$$C_*(-) \otimes C_*(-) \xrightarrow{\nabla} C_*(- \times -),$$

- *lax comonoidal, via the Alexander-Whitney map*

$$C_*(- \times -) \xrightarrow{f} C_*(-) \otimes C_*(-),$$

- (∇, f) *satisies braiding and unitality axioms.*

Algebraic consequences

- C_*X is a chain coalgebra, for all simplicial sets X.
- C_*G is a chain Hopf algebra, for all simplicial groups G.
- More generally, if X admits a G-action, then C_*X is a C_*G-module coalgebra.

Szczarba's Theorem

Theorem (Szczarba)

For any $X \in \mathbf{sSet}_0$, there is a natural twisting cochain

$$Sz_X : C_*X \to C_*\mathscr{G}X$$

inducing a natural morphism of chain algebras

$$\alpha_X : \Omega C_*X \to C_*\mathscr{G}X,$$

which is a quasi-isomorphism if X is 1-reduced.

Corollary

There is a natural transformation

$$\alpha : \Omega \circ C_* \to C_* \circ \mathscr{G}.$$

Extending Szczarba's equivalence

Theorem (H.-Parent-Scott)

If $g : X \to Y$ is a simplicial map, where Y is 1-reduced and both X and Y are of finite type, then there is a natural commuting diagram of chain complexes

$$\begin{array}{ccccc} \Omega C_* Y & \longrightarrow & C_* X \square_{C_* Y} \mathscr{P} C_* Y & \longrightarrow & C_* X \\ \sim \downarrow \alpha_Y & & \sim \downarrow \widehat{\alpha}_g & & = \downarrow \\ C_* \mathscr{G} Y & \longrightarrow & C_*(X \underset{Y}{\times} \mathscr{P} Y) & \longrightarrow & C_* X. \end{array}$$

Corollary

The natural transformation $\alpha : \Omega \circ C_* \to C_* \circ \mathscr{G}$ extends to

$$\widehat{\alpha} : \mathscr{P} \circ C_* \to C_* \circ \mathscr{P}.$$

The motivating question

What is an appropriate categorical set-up that captures the twisted constructions on $\mathbf{sSet}_0$ and $\mathbf{Ch}_{\Bbbk}$, as mediated by the normalized chains functor?

Why do we care?

- In $\mathbf{Ch}_{\Bbbk}$, twisted tensor products give us resolutions, the key tools for computation in homological algebra.
- In $\mathbf{sSet}_0$, twisted cartesian products model all fibrations, and every simplicial morphism is homotopy-equivalent to a fibration.
- Understanding the relationship between $\mathscr{P}C_*$ and $C_*\mathscr{P}$ and between $\mathscr{E}C_*$ and $C_*\mathscr{E}$ enables us to develop a general method of constructing chain complex models of simplicial sets: given models of $\mathscr{E}G$ and $\mathscr{P}X$, build up models any other space by appropriate equalizer and coequalizer constructions.

Applications thus far

- [H.- Scott] Proof of the existence of such a twisting structure on the category of symmetric sequences of “nice” chain complexes, leading to a characterization of strongly homotopy morphisms of $\mathcal{O}$-algebras and of $\mathcal{O}_\infty$-algebras, when $\mathcal{O}$ is a Koszul operad.
- [Farjoun-H.] Elaboration of a theory of “homotopy-normal” morphisms of monoids in a twisted monoidal category with an appropriately compatible notion of weak equivalence.

The category of mixed modules

Let $(\mathbf{M}, \otimes, I)$ be a monoidal category that is twistable: formation of tensor products over monoids commutes with formation of cotensor products over comonoids. Define a category **Mix** by

$$\mathrm{Ob}\,\mathbf{Mix} = \mathrm{Ob}\,\mathbf{Mon} \cup \mathrm{Ob}\,\mathbf{Comon}$$

and

$$\mathbf{Mix}(X, Y) = {}_X\mathbf{Mix}_Y / \cong,$$

where composition in **Mix** is given by tensoring over monoids and cotensoring over comonoids.

Useful functors

- $I : \mathbf{Mon} \to \mathbf{Mix}$ and $\widetilde{I} : \mathbf{Mon}^{op} \to \mathbf{Mix}$:

$$I(A) = A = \widetilde{I}(A)$$

and for all $f : A \to A'$,

$$I(f) = {}_fA' \in \mathbf{Mix}(A, A') \text{ and } \widetilde{I}(f) = A'_f \in \mathbf{Mix}(A', A).$$

- $J : \mathbf{Comon} \to \mathbf{Mix}$ and $\widetilde{J} : \mathbf{Comon}^{op} \to \mathbf{Mix}$:

$$J(C) = C = \widetilde{J}(C),$$

and for all $g : C \to C'$,

$$J(g) = C_g \in \mathbf{Mix}(C, C') \text{ and } \widetilde{J}(g) = {}_gC \in \mathbf{Mix}(C', C).$$

The data of a right twisting structure

- A functor $\mathscr{B} : \mathbf{Mon} \to \mathbf{Comon}$
- Natural transformations $\mathscr{E} : J \circ \mathscr{B} \Rightarrow J$ and $\widetilde{\mathscr{E}} : \widetilde{J} \Rightarrow \widetilde{J} \circ \mathscr{B}$ of functors from **Mon** to **Mix**
- Natural morphisms

$$\delta_A : \mathscr{B}A \to \mathscr{E}A \underset{A}{\otimes} \widetilde{\mathscr{E}}A$$

of $\mathscr{B}A$-bicomodules and

$$\mu_A : \widetilde{\mathscr{E}}A \underset{\mathscr{B}A}{\square} \mathscr{E}A \to A$$

of A-bimodules.

The compatibility condition

The data $(\mathbf{M}, \mathscr{B}, \mathscr{E}, \widetilde{\mathscr{E}}, \delta, \mu)$ form a right twisting structure if

$$\begin{array}{ccc}
\widetilde{\mathscr{E}}A \cong \widetilde{\mathscr{E}}A \underset{\mathscr{B}A}{\square} \mathscr{B}A & \xrightarrow{\widetilde{\mathscr{E}}A \underset{\mathscr{B}A}{\square} \delta_A} & \widetilde{\mathscr{E}}A \underset{\mathscr{B}A}{\square} \mathscr{E}A \underset{A}{\otimes} \widetilde{\mathscr{E}}A \\
& \searrow^{=} & \downarrow{\scriptstyle \mu_A \underset{A}{\otimes} \widetilde{\mathscr{E}}A} \\
& & \widetilde{\mathscr{E}}A
\end{array}$$

and

$$\begin{array}{ccc}
\mathscr{E}A \cong \mathscr{B}A \underset{\mathscr{B}A}{\square} \mathscr{E}A & \xrightarrow{\delta_A \underset{\mathscr{B}A}{\square} \mathscr{E}A} & \mathscr{E}A \underset{A}{\otimes} \widetilde{\mathscr{E}}A \underset{\mathscr{B}A}{\square} \mathscr{E}A \\
& \searrow^{=} & \downarrow{\scriptstyle \mathscr{E}A \underset{A}{\otimes} \mu_A} \\
& & \mathscr{E}A
\end{array}$$

commute.

The dual situation

There is a strictly dual notion of left twisting structures $(\Omega, \mathscr{P}, \widetilde{\mathscr{P}}, \mu, \delta)$ on twistable model categories.

A right twisting structure $(\mathbf{M}, \mathscr{B}, \mathscr{E}, \widetilde{\mathscr{E}}, \delta, \mu)$ and an adjoint pair of functors $(\Omega, \mathscr{B})$ together give rise to a left twisting structure on **M**. The dual result holds as well.

A twisting structure on a twistable model category **M** consists of compatible left and right twisting structures on **M**. We say then that **M** is twisted.

The big picture

The data and properties of a twisting structures can be neatly summarized in terms of the existence of an adjunction of bifibrations.

Preservation of twisting structures

Theorem (H.-Lack)

Let $\mathbf{M}$ *and* $\mathbf{M}'$ *be twisted monoidal categories, and let* $\Phi : \mathbf{M} \to \mathbf{M}'$ *be a normal bilax monoidal functor. If there exist suitably compatible natural transformations*

$$\Phi(\mathscr{E}A \otimes_A M) \to \mathscr{E}\Phi A \otimes_{\Phi A} \Phi M$$

and

$$\Phi N \Box_{\Phi C} \mathcal{P}\Phi C \to \Phi(N \Box_C \mathcal{P}C),$$

then Φ *induces a morphism of twisted monoidal categories.*

A bundle perspective

Let **M** be a twisted monoidal category.

Note that for any monoid A,

$$\mathscr{E}A \in {}_{\mathscr{B}A}\mathbf{Mix}_A \quad \text{and} \quad \widetilde{\mathscr{E}}A \in {}_A\mathbf{Mix}_{\mathscr{B}A}.$$

We think of $\mathscr{E}A$ and $\widetilde{\mathscr{E}}A$ as the total spaces of the "universal right A-bundle" and the "universal left A-bundle" in **M**, respectively.

Dually, for any comonoid C, we view $\mathscr{P}C$ and $\widetilde{\mathscr{P}}C$ as based path spaces over C.

Classifying morphisms and induced bundles

Let $g : C \to \mathscr{B}A$ be a morphism of comonoids in a twisted monoidal category.

Given a right C-comodule V and a left A-module W,

$$V \otimes_g W := V \underset{\mathscr{B}A}{\square} \mathscr{E}A \underset{A}{\otimes} W,$$

where V considered as a right BA-comodule via g.

If X is a right A-module and Y is a left C-comodule, then

$$X \otimes_g Y := X \underset{A}{\otimes} \tilde{\mathscr{E}}A \underset{\mathscr{B}A}{\square} Y,$$

where Y considered as a left BA-comodule via g.

A useful adjunction

Theorem (H.-Scott)

Let $g : C \to \mathscr{B}A$ be a comonoid morphism in a twisted monoidal category. If

$$g_* : {}_C\mathbf{Comod} \to {}_A\mathbf{Mod} : N \mapsto A \otimes_g N$$

and

$$g^* : {}_A\mathbf{Mod} \to {}_C\mathbf{Comod} : M \mapsto C \otimes_g M,$$

then

$$g_* : {}_C\mathbf{Comod} \rightleftarrows {}_A\mathbf{Mod} : g^*$$

is an adjoint pair.

Standard constructions

Let $g : C \to \mathscr{B}A$ be a comonoid morphism.

The standard construction on g:

$$K(g) := A \otimes_g C \otimes_g A.$$

Proposition (H.-Scott)

The standard construction $K(g)$ is naturally an A-co-ring, i.e., a comonoid in the category of A-bimodules.

Monoidal structures on symmetric sequences

Let $(\mathbf{M}, \otimes, I)$ be a cocomplete, symmetric monoidal category. Let $\mathcal{X}, \mathcal{Y} \in \mathbf{M}^{\Sigma}$.

The composition product of $\mathcal{X}$ and $\mathcal{Y}$:

$$(\mathcal{X} \circ \mathcal{Y})(n) = \bigoplus_{k \geq 0, \vec{n} \in J_{n,k}} \mathcal{X}(k) \otimes_{\Sigma_m} \mathcal{Y}(n_1) \otimes \cdots \otimes \mathcal{Y}(n_k) \otimes_{\Sigma_{\vec{n}}} I[\Sigma_n],$$

where $J_{n,k} = \{\vec{n} = (n_1, .., n_k) \mid \sum_i n_i = n, n_i \in \mathbb{N}\ \forall i\}$.

Theorem

$(\mathbf{M}^{\Sigma}, \circ, \mathcal{J})$ *is a monoidal category, where*

$$\mathcal{J}(n) = \begin{cases} I & n = 1, \\ \emptyset & \textit{else.} \end{cases}$$

The algebra of symmetric sequences

A monoid in $(\mathbf{M}^{\Sigma}, \circ, \mathcal{J})$ is an operad in **M**.

If $\mathcal{P}$ is an operad in **M**, a $\mathcal{P}$-algebra is a left $\mathcal{P}$-module, concentrated in level 0.

The algebra of symmetric sequences

A comonoid in $(\mathbf{M}^{\Sigma}, \circ, \mathcal{J})$ is a cooperad in **M**.

If $\mathcal{P}$ is a cooperad in **M**, a $\mathcal{P}$-coalgebra is a left $\mathcal{P}$-comodule, concentrated in level 0.

The case of chain complexes

Let $\mathbb{k}$ be a commutative ring.

Let $\mathbf{dgFGP}_{\mathbb{k}}$ denote the category of chain complexes that are degreewise finitely generated, projective R-modules.

Theorem (H.-Scott)

The monoidal category $(\mathbf{dgFGP}_{\mathbb{k}}^{\Sigma}, \circ, \mathcal{J})$ *admits a twisting structure based on the operadic bar* $\mathbf{B}$ *and cobar* $\mathbf{\Omega}$ *constructions of Ginzburg and Kapranov.*

Generalized bar/cobar-adjunctions

Let $g : \mathcal{Q} \to \mathbf{B}\mathcal{P}$ be a cooperad morphism in $\mathbf{dgFGP}_R$.

The g-cobar construction

$$\Omega_g = g_* : {}_{\mathcal{Q}}\mathbf{Comod} \to {}_{\mathcal{P}}\mathbf{Mod} : \mathcal{M} \mapsto \mathcal{P} \circ_g \mathcal{M}$$

and the g-bar construction:

$$\mathscr{B}_g = g^* : {}_{\mathcal{P}}\mathbf{Mod} \to {}_{\mathcal{Q}}\mathbf{Comod} : \mathcal{N} \mapsto \mathcal{Q} \circ_g \mathcal{N}$$

restrict to define

$$\Omega_g : \mathcal{Q}\text{-}\mathbf{Coalg} \to \mathcal{P}\text{-}\mathbf{Alg} \quad \text{and} \quad \mathscr{B}_g : \mathcal{P}\text{-}\mathbf{Alg} \to \mathcal{Q}\text{-}\mathbf{Coalg}.$$

Proposition

For any $g : \mathcal{Q} \to \mathbf{B}\mathcal{P}$ as above, Ω_g is left adjoint to $\mathscr{B}_g$.

Bar/cobar-adjunctions of Koszul operads

Let $\mathcal{P}$ be a quadratic operad, and let $\kappa_{\mathcal{P}} : \mathcal{P}^{\perp} \to \mathbf{B}\mathcal{P}$ be the canonical inclusion.

- If $\mathcal{P}$ is Koszul (=$\kappa_{\mathcal{P}}$ is a quasi-isomorphism), then

$$\Omega_{\kappa_{\mathcal{P}}} = \Omega_{\mathcal{P}^{\perp}} : \mathcal{P}^{\perp}\text{-}\mathbf{Coalg} \rightleftarrows \mathcal{P}\text{-}\mathbf{Alg} : \mathscr{B}_{\kappa_{\mathcal{P}}} = \mathscr{B}_{\mathcal{P}},$$

  the bar and cobar constructions of Getzler and Jones.
- The previous proposition generalizes the result of Getzler and Jones that $\Omega_{\mathcal{P}^{\perp}}$ is left adjoint to $\mathscr{B}_{\mathcal{P}}$.

The Koszul construction

If $\mathcal{P}$ is a Koszul operad, then

$$K(\mathcal{P}) := K(\kappa_{\mathcal{P}})$$

is Fresse's two-sided Koszul resolution of $\mathcal{P}$.

Recall that $K(\mathcal{P})$ is necessarily a $\mathcal{P}$-co-ring, since it is a standard construction on a cooperad morphism.

Let

$$\mathsf{K}_{\mathcal{P}} : \mathcal{P}\text{-}\mathbf{Alg} \to \mathcal{P}\text{-}\mathbf{Alg}$$

denote the comonad with underlying endofunctor $K(\mathcal{P}) \circ_{\mathcal{P}} -$.

Let ${}_{\mathsf{K}_{\mathcal{P}}}\mathcal{P}$-**Alg** denote the associated coKleisli category.

Characterizing the category $\mathcal{P}$-$\mathbf{Alg}_{sh}$

Let $\mathcal{P}$ be a Koszul operad.

The category $\mathcal{P}$-$\mathbf{Alg}_{sh}$ of $\mathcal{P}$-algebras and strongly homotopy morphisms:

$$\mathrm{Ob}\, \mathcal{P}\text{-}\mathbf{Alg}_{sh} = \mathrm{Ob}\, \mathcal{P}\text{-}\mathbf{Alg}$$
$$\mathcal{P}\text{-}\mathbf{Alg}_{sh}(A, A') = \mathcal{P}^{\perp}\text{-}\mathbf{Coalg}(\mathscr{B}_{\mathcal{P}}A, \mathscr{B}_{\mathcal{P}}A')$$

Theorem (H.-Scott)

There is a natural isomorphism of categories

$$\mathcal{P}\text{-}\mathbf{Alg}_{sh} \xrightarrow{\cong} {}_{\mathsf{K}_{\mathcal{P}}}\mathcal{P}\text{-}\mathbf{Alg}.$$

This result can be generalized to characterize the category of $\mathcal{P}_{\infty}$-algebras and their strongly homotopy morphisms as the coKleisli category associated to the comonad arising from an appropriate standard construction.