Twisting Structures

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Joint work with...

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- Jonathan Scott (application to characterizing strongly homotopy maps)
- Emmanuel Dror Farjoun (application to homotopy (co)normality)
Outline

1. Motivation
   - Twisting cochains
   - Twisting functions
   - From twisting functions to twisting cochains

2. The categorical framework
   - General twisting structures
   - Bundle theory

3. Application: strong homotopy
   - Operads and their algebras
   - Twisting structures on symmetric sequences
   - The case of Koszul operads
Let \( k \) be a commutative ring.

- \( \text{Coalg}_k \): the category of 1-connected, coaugmented chain coalgebras over \( k \).

- \( \text{Alg}_k \): the category of connected, augmented chain algebras over \( k \).
The adjunction

\[ \Omega : \text{Coalg}_k \leftrightarrow \text{Alg}_k : \mathcal{B} \]

is an adjoint pair with unit

\[ \eta : \text{Id} \to \mathcal{B}\Omega \]

and counit

\[ \varepsilon : \Omega\mathcal{B} \to \text{Id}, \]

which are objectwise quasi-isomorphisms.
Twisting cochains

Given

- $(C, d_C) \in \text{Coalg}_k$ with comultiplication $\Delta$,
- $(A, d_A) \in \text{Alg}_k$ with multiplication $\mu$,

a linear map of degree $-1$

$$t : C \to A$$

is a twisting cochain if

$$d_A t + td_C = \mu(t \otimes t) \Delta.$$
The universal twisting cochain on $C \in \text{Coalg}_{k}$:

$$t_{\Omega} : C \to \Omega C.$$ 

All twisting cochains $t : C \to A$ factor uniquely through $t_{\Omega}$.

$$ \forall t \quad \exists! \alpha_{t} $$

$$ C \xrightarrow{t_{\Omega}} \Omega C \quad \downarrow \alpha_{t} \quad \downarrow \forall t \quad \downarrow A $$
The couniversal twisting cochain to $A \in \text{Alg}_k$:

$$t_B : BA \rightarrow A.$$ 

All twisting cochains $t : C \rightarrow A$ factor uniquely through $t_B$.

$$\exists! \beta_t \downarrow \forall t \downarrow t_B \downarrow BA \rightarrow A$$
Twisted tensor products

Given

- $t : C \to A$, a twisting cochain;
- $M$ a right $C$-comodule, with $C$-coaction $\rho : M \to M \otimes C$;
- $N$ a left $A$-module, with $A$-action $\lambda : A \otimes N \to N$.

The **twisted tensor product** of $M$ and $N$ over $t$ is a chain complex

$$M \otimes_t N = (M \otimes N, D_t),$$

where

$$D_t = d_M \otimes N + M \otimes d_N - (M \otimes \lambda)(M \otimes t \otimes N)(\rho \otimes N).$$
The universal bundles

- \( t_\Omega : C \to \Omega C \implies PC = C \otimes_{t_\Omega} \Omega C \simeq k \).
- \( t_B : BA \to A \implies EA = BA \otimes_{t_B} A \simeq k \).

If \( t : C \to A \) is a twisting cochain, \( M \) is a right \( C \)-comodule and \( N \) is a left \( A \)-module, then

\[
\begin{align*}
M \square_C PC \otimes_{\Omega C} N & \sim \\
M \otimes_{t} N & \\
M \square_B BA \otimes_{A} N & \sim
\end{align*}
\]

Every twisted tensor product can be constructed from the universal bundles, by equalizing coactions and coequalizing actions.
The framework

- \( \text{sSet}_0 \) = the category of reduced simplicial sets.
- \( \text{sGr} \) = the category of simplicial groups.
The adjunction

\[ G : \text{sSet}_0 \rightleftarrows \text{sGr} : \overline{W} \]

is an adjoint pair with unit

\[ \eta : \text{Id} \to \overline{W} G \]

and counit

\[ \varepsilon : G \overline{W} \to \text{Id}, \]

which are objectwise weak equivalences.
Twisting functions

Given
- $X$, a simplicial set, and
- $G$ a simplicial group,
a degree $-1$ map of graded sets

$$\tau : X \to G$$

is a **twisting function** if

\[
\begin{align*}
d_0 \tau (x) &= (\tau (d_0 x))^{-1} \tau (d_1 x) \\
d_i \tau (x) &= \tau (d_{i+1} x), \quad i > 0 \\
s_i \tau (x) &= \tau (s_{i+1} x), \quad i \geq 0 \\
\tau (s_0 x) &= e
\end{align*}
\]

for all $x \in X$. 
The universal twisting function on $X \in s\text{Set}_0$:

$$\tau^G : X \to \mathcal{G} X.$$ 

All twisting functions $\tau : X \to G$ factor uniquely through $\tau^G$.

\[ X \xrightarrow{\tau^G} \mathcal{G} X \]

\[ \forall \tau \xrightarrow{} \exists! \alpha_{\tau} \]

\[ \downarrow \]

\[ G \]
The couniversal twisting function to $G \in \text{sGr}$:

$$\tau_{\overline{W}} : \overline{W} G \to G.$$ 

All twisting functions $\tau : X \to G$ factor uniquely through $\tau_{\overline{W}}$. 

\[
\begin{array}{ccc}
X & \xrightarrow{\exists! \beta_\tau} & \overline{W} G \\
& \searrow & \downarrow \forall \tau \\
& \downarrow t_{\overline{W}} & \\
\overline{W} G & \xrightarrow{t_{\overline{W}}} & G
\end{array}
\]
Twisted cartesian products

Given

- $\tau : X \to G$, a twisting function,
- a left action of $G$ on a simplicial set $Z$,
- a simplicial map $f : Y \to X$ ($\iff$ a right coaction of $(X, \Delta)$ on $Y$)
Twisted cartesian products

Given

- $\tau : X \rightarrow G$, a twisting function,
- a left action of $G$ on a simplicial set $Z$,
- a simplicial map $f : Y \rightarrow X$.

The **twisted cartesian product** of $Y$ and $Z$, denoted $Y \times_\tau Z$, is a simplicial set such that

$$(Y \times_\tau Z)_n = Y_n \times Z_n,$$

with faces and degeneracies given by

- $d_0(y, z) = (d_0y, \tau(f(x)) \cdot d_0z)$
- $d_i(y, z) = (d_iy, d_iz)$, $i > 0$
- $s_i(y, z) = (s_iy, s_iz)$, $i \geq 0$. 

The universal bundles

- $\tau^G : X \to GX \implies \mathcal{P}X = X \times_{\tau^G} GX \simeq \ast$.
- $\tau^W : W G \to G \implies \mathcal{E}G = WG \times_{\tau^W} G \simeq \ast$.

If $\tau : X \to G$ is a twisting cochain, $Y \to X$ is a simplicial map and $Z$ is a left $G$-module, then

$$Y \times \mathcal{P}X \times_{\tau^G} XZ \underset{\tau}{\to} Y \times \tau Z \underset{\tau}{\leftarrow} Y \times \mathcal{E}G \times_{\tau^G} GZ.$$

Every twisted cartesian product can be constructed from the universal bundles by pullbacks and by coequalizing group actions.
The normalized chains functor

Theorem (Aguiar-Mahajan)

The normalized chains functor

\[ C_* : \text{sSet}_0 \rightarrow \text{Ch}_k \]

is a normal bilax monoidal functor:

- **lax monoidal, via the Eilenberg-Zilber map**
  \[ C_*(-) \otimes C_*(-) \xrightarrow{\nabla} C_*(- \times -), \]

- **lax comonoidal, via the Alexander-Whitney map**
  \[ C_*(- \times -) \xrightarrow{f} C_*(-) \otimes C_*(-), \]

- \((\nabla, f)\) satisfies braiding and unitality axioms.
Algebraic consequences

- $C_* X$ is a chain coalgebra, for all simplicial sets $X$.
- $C_* G$ is a chain Hopf algebra, for all simplicial groups $G$.
- More generally, if $X$ admits a $G$-action, then $C_* X$ is a $C_* G$-module coalgebra.
Szczarba’s Theorem

Theorem (Szczarba)

For any $X \in \mathbf{sSet}_0$, there is a natural twisting cochain

$$\text{Sz}_X : \text{C}_* X \to \text{C}_* \mathcal{G} X$$

inducing a natural morphism of chain algebras

$$\alpha_X : \Omega \text{C}_* X \to \text{C}_* \mathcal{G} X,$$

which is a quasi-isomorphism if $X$ is 1-reduced.

Corollary

There is a natural transformation

$$\alpha : \Omega \circ \text{C}_* \to \text{C}_* \circ \mathcal{G}.$$
Extending Szczarba’s equivalence

**Theorem (H.-Parent-Scott)**

If \( g : X \to Y \) is a simplicial map, where \( Y \) is 1-reduced and both \( X \) and \( Y \) are of finite type, then there is a natural commuting diagram of chain complexes

\[
\begin{align*}
\Omega C_* Y & \to C_* X \sqcup_{C_* Y} P C_* Y \to C_* X \\
\sim \downarrow \alpha_Y & \sim \downarrow \hat{\alpha}_g \\
C_* G Y & \to C_* (X \times_Y P Y) \to C_* X.
\end{align*}
\]

**Corollary**

The natural transformation \( \alpha : \Omega \circ C_* \to C_* \circ G \) extends to

\[
\hat{\alpha} : P \circ C_* \to C_* \circ P.
\]
What is an appropriate categorical set-up that captures the twisted constructions on $\mathbf{sSet}_0$ and $\mathbf{Ch}_k$, as mediated by the normalized chains functor?
Why do we care?

- In $\mathbf{Ch}_k$, twisted tensor products give us resolutions, the key tools for computation in homological algebra.
- In $\mathbf{sSet}_0$, twisted cartesian products model all fibrations, and every simplicial morphism is homotopy-equivalent to a fibration.
- Understanding the relationship between $\mathcal{P}C_*$ and $C_*\mathcal{P}$ and between $\mathcal{E}C_*$ and $C_*\mathcal{E}$ enables us to develop a general method of constructing chain complex models of simplicial sets: given models of $\mathcal{E}G$ and $\mathcal{P}X$, build up models any other space by appropriate equalizer and coequalizer constructions.
Applications thus far

- [H.- Scott] Proof of the existence of such a twisting structure on the category of symmetric sequences of “nice” chain complexes, leading to a characterization of strongly homotopy morphisms of $\mathcal{O}$-algebras and of $\mathcal{O}_\infty$-algebras, when $\mathcal{O}$ is a Koszul operad.

The category of mixed modules

Let \((\mathcal{M}, \otimes, I)\) be a monoidal category that is twistable: formation of tensor products over monoids commutes with formation of cotensor products over comonoids. Define a category \(\text{Mix}\) by

\[
\text{Ob Mix} = \text{Ob Mon} \cup \text{Ob Comon}
\]

and

\[
\text{Mix}(X, Y) = X^\text{Mix}_Y / \cong,
\]

where composition in \(\text{Mix}\) is given by tensoring over monoids and cotensoring over comonoids.
Useful functors

- $I : \text{Mon} \to \text{Mix}$ and $\tilde{I} : \text{Mon}^{\text{op}} \to \text{Mix}$:

$$I(A) = A = \tilde{I}(A)$$

and for all $f : A \to A'$,

$$I(f) = fA' \in \text{Mix}(A, A') \text{ and } \tilde{I}(f) = A'_f \in \text{Mix}(A', A).$$

- $J : \text{Comon} \to \text{Mix}$ and $\tilde{J} : \text{Comon}^{\text{op}} \to \text{Mix}$:

$$J(C) = C = \tilde{J}(C),$$

and for all $g : C \to C'$,

$$J(g) = C_g \in \text{Mix}(C, C') \text{ and } \tilde{J}(g) = gC \in \text{Mix}(C', C).$$
The data of a right twisting structure

- A functor $\mathcal{B} : \text{Mon} \to \text{Comon}$
- Natural transformations $\mathcal{E} : J \circ \mathcal{B} \Rightarrow J$ and $\widetilde{\mathcal{E}} : \tilde{J} \Rightarrow \tilde{J} \circ \mathcal{B}$ of functors from $\text{Mon}$ to $\text{Mix}$
- Natural morphisms

$$\delta_A : \mathcal{B}A \to \mathcal{E}A \otimes_{A} \mathcal{E}A$$

of $\mathcal{B}A$-bicomodules and

$$\mu_A : \mathcal{E}A \Box_{\mathcal{B}A} \mathcal{E}A \to A$$

of $A$-bimodules.
The compatibility condition

The data \((M, B, E, \tilde{E}, \delta, \mu)\) form a right twisting structure if

\[
\tilde{E}A \cong \tilde{E}A \Box BA \xrightarrow{\tilde{E}A \Box \delta A} \tilde{E}A \Box E A \otimes \tilde{E}A \xrightarrow{\mu A \otimes \tilde{E}A} \tilde{E}A
\]

and

\[
E A \cong BA \Box E A \xrightarrow{\delta A \Box E A} E A \otimes \tilde{E}A \Box E A \xrightarrow{E A \otimes \mu A} E A
\]

commute.
The dual situation

There is a strictly dual notion of left twisting structures \((\Omega, \mathcal{P}, \widetilde{\mathcal{P}}, \mu, \delta)\) on twistable model categories.

A right twisting structure \((\mathcal{M}, \mathcal{B}, \mathcal{E}, \widetilde{\mathcal{E}}, \delta, \mu)\) and an adjoint pair of functors \((\Omega, \mathcal{B})\) together give rise to a left twisting structure on \(\mathcal{M}\). The dual result holds as well.

A twisting structure on a twistable model category \(\mathcal{M}\) consists of compatible left and right twisting structures on \(\mathcal{M}\). We say then that \(\mathcal{M}\) is twisted.
The data and properties of a twisting structures can be neatly summarized in terms of the existence of an adjunction of bifibrations.
Preservation of twisting structures

Theorem (H.-Lack)

Let $M$ and $M'$ be twisted monoidal categories, and let $\Phi : M \to M'$ be a normal bilax monoidal functor. If there exist suitably compatible natural transformations

$$\Phi (\mathcal{E} A \otimes_A M) \to \mathcal{E} \Phi A \otimes_{\Phi A} \Phi M$$

and

$$\Phi N \Box_{\Phi C} \mathcal{P} \Phi C \to \Phi (N \Box_{C} \mathcal{P} C),$$

then $\Phi$ induces a morphism of twisted monoidal categories.
A bundle perspective

Let $\mathcal{M}$ be a twisted monoidal category.

Note that for any monoid $A$,

$$\mathcal{E} A \in B^A \text{Mix}_A \quad \text{and} \quad \tilde{\mathcal{E}} A \in A \text{Mix}_B.$$

We think of $\mathcal{E} A$ and $\tilde{\mathcal{E}} A$ as the total spaces of the “universal right $A$-bundle” and the “universal left $A$-bundle” in $\mathcal{M}$, respectively.

Dually, for any comonoid $C$, we view $\mathcal{P} C$ and $\tilde{\mathcal{P}} C$ as based path spaces over $C$. 
Classifying morphisms and induced bundles

Let \( g : C \to BA \) be a morphism of comonoids in a twisted monoidal category.

Given a right \( C \)-comodule \( V \) and a left \( A \)-module \( W \),

\[
V \otimes g W := V \otimes_B E A \otimes_A W,
\]

where \( V \) considered as a right \( BA \)-comodule via \( g \).

If \( X \) is a right \( A \)-module and \( Y \) is a left \( C \)-comodule, then

\[
X \otimes g Y := X \otimes_A E A \otimes_B Y,
\]

where \( Y \) considered as a left \( BA \)-comodule via \( g \).
A useful adjunction

<table>
<thead>
<tr>
<th>Theorem (H.-Scott)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $g : C \to \mathcal{B}A$ be a comonoid morphism in a twisted monoidal category. If</td>
</tr>
<tr>
<td>$g_* : C\text{Comod} \to A\text{Mod} : N \mapsto A \otimes g N$</td>
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<tr>
<td>and</td>
</tr>
<tr>
<td>$g^* : A\text{Mod} \to C\text{Comod} : M \mapsto C \otimes g M$,</td>
</tr>
<tr>
<td>then</td>
</tr>
<tr>
<td>$g_* : C\text{Comod} \rightleftarrows A\text{Mod} : g^*$</td>
</tr>
<tr>
<td>is an adjoint pair.</td>
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</tbody>
</table>
Standard constructions

Let $g : C \to \mathcal{B}A$ be a comonoid morphism.

The standard construction on $g$:

$$K(g) := A \otimes_g C \otimes_g A.$$

**Proposition (H.-Scott)**

*The standard construction $K(g)$ is naturally an $A$-co-ring, i.e., a comonoid in the category of $A$-bimodules.*
Monoidal structures on symmetric sequences

Let \((\mathcal{M}, \otimes, I)\) be a cocomplete, symmetric monoidal category. Let \(\mathcal{X}, \mathcal{Y} \in \mathcal{M}^\Sigma\).

The composition product of \(\mathcal{X}\) and \(\mathcal{Y}\):

\[
(\mathcal{X} \circ \mathcal{Y})(n) = \bigoplus_{k \geq 0, \vec{n} \in J_{n,k}} \mathcal{X}(k) \otimes \sum_m \mathcal{Y}(n_1) \otimes \cdots \otimes \mathcal{Y}(n_k) \otimes \sum \vec{n} I[\Sigma n],
\]

where \(J_{n,k} = \{ \vec{n} = (n_1, \ldots, n_k) \mid \sum_i n_i = n, n_i \in \mathbb{N} \, \forall \, i \}\).

**Theorem**

\((\mathcal{M}^\Sigma, \circ, \mathcal{J})\) is a monoidal category, where

\[
\mathcal{J}(n) = \begin{cases} 
I & n = 1, \\
\emptyset & \text{else.}
\end{cases}
\]
The algebra of symmetric sequences

A monoid in \((\mathbf{M}^\Sigma, \circ, J)\) is an operad in \(\mathbf{M}\).

If \(\mathcal{P}\) is an operad in \(\mathbf{M}\), a \(\mathcal{P}\)-algebra is a left \(\mathcal{P}\)-module, concentrated in level 0.
A comonoid in \((M^\Sigma, \circ, J)\) is a cooperad in \(M\).

If \(P\) is a cooperad in \(M\), a \(P\)-coalgebra is a left \(P\)-comodule, concentrated in level 0.
The case of chain complexes

Let $k$ be a commutative ring.

Let $\text{dgFGP}_k$ denote the category of chain complexes that are degreewise finitely generated, projective $R$-modules.

Theorem (H.-Scott)

The monoidal category $(\text{dgFGP}_k^\Sigma, \circ, \mathcal{J})$ admits a twisting structure based on the operadic bar $\mathcal{B}$ and cobar $\Omega$ constructions of Ginzburg and Kapranov.
Generalized bar/cobar-adjunctions

Let $g : \mathcal{Q} \to \mathcal{B}\mathcal{P}$ be a cooperad morphism in $\text{dgFGP}_R$.

The $g$-cobar construction

$$\Omega_g = g^* : \mathcal{Q}\text{Comod} \to \mathcal{P}\text{Mod} : \mathcal{M} \mapsto \mathcal{P} \circ g \mathcal{M}$$

and the $g$-bar construction:

$$\mathcal{B}_g = g^* : \mathcal{P}\text{Mod} \to \mathcal{Q}\text{Comod} : \mathcal{N} \mapsto \mathcal{Q} \circ g \mathcal{N}$$

restrict to define

$$\Omega_g : \mathcal{Q}\text{-Coalg} \to \mathcal{P}\text{-Alg} \quad \text{and} \quad \mathcal{B}_g : \mathcal{P}\text{-Alg} \to \mathcal{Q}\text{-Coalg}.$$  

**Proposition**

*For any $g : \mathcal{Q} \to \mathcal{B}\mathcal{P}$ as above, $\Omega_g$ is left adjoint to $\mathcal{B}_g$.***
Let $\mathcal{P}$ be a quadratic operad, and let $\kappa_{\mathcal{P}} : \mathcal{P}^\perp \to B\mathcal{P}$ be the canonical inclusion.

- If $\mathcal{P}$ is Koszul ($=\kappa_{\mathcal{P}}$ is a quasi-isomorphism), then

$$\Omega_{\kappa_{\mathcal{P}}} = \Omega_{\mathcal{P}^\perp} : \mathcal{P}^\perp \text{-Coalg} \rightleftarrows \mathcal{P} \text{-Alg} : B_{\kappa_{\mathcal{P}}} = B\mathcal{P},$$

the bar and cobar constructions of Getzler and Jones.

- The previous proposition generalizes the result of Getzler and Jones that $\Omega_{\mathcal{P}^\perp}$ is left adjoint to $B\mathcal{P}$. 
The Koszul construction

If $\mathcal{P}$ is a Koszul operad, then

$$K(\mathcal{P}) := K(\kappa_{\mathcal{P}})$$

is Fresse's two-sided Koszul resolution of $\mathcal{P}$.

Recall that $K(\mathcal{P})$ is necessarily a $\mathcal{P}$-co-ring, since it is a standard construction on a cooperad morphism.

Let

$$K_{\mathcal{P}} : \mathcal{P}\text{-Alg} \rightarrow \mathcal{P}\text{-Alg}$$

denote the comonad with underlying endofunctor $K(\mathcal{P}) \circ_{\mathcal{P}} -$.

Let $K_{\mathcal{P}} \mathcal{P}\text{-Alg}$ denote the associated coKleisli category.
Characterizing the category $\mathcal{P} \text{-Alg}_{sh}$

Let $\mathcal{P}$ be a Koszul operad.

The category $\mathcal{P} \text{-Alg}_{sh}$ of $\mathcal{P}$-algebras and strongly homotopy morphisms:

$$\text{Ob } \mathcal{P} \text{-Alg}_{sh} = \text{Ob } \mathcal{P} \text{-Alg}$$

$$\mathcal{P} \text{-Alg}_{sh}(A, A') = \mathcal{P} \text{-Coalg}(\mathcal{B}_\mathcal{P} A, \mathcal{B}_\mathcal{P} A')$$

**Theorem (H.-Scott)**

*There is a natural isomorphism of categories*

$$\mathcal{P} \text{-Alg}_{sh} \cong \mathcal{K}_\mathcal{P} \mathcal{P} \text{-Alg}.$$

This result can be generalized to characterize the category of $\mathcal{P}_\infty$-algebras and their strongly homotopy morphisms as the coKleisli category associated to the comonad arising from an appropriate standard construction.