Power Maps in Algebra and Topology

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Outline

Conventions

- R is a PID
- $\mathbf{Ch}_R$ is the category of chain complexes of R-modules that are R-free as graded R-modules.
- $\mathbf{Alg}_R$ is the category of augmented, associative dg algebras over R.
- $\mathbf{Coalg}_R$ is the category of coaugmented, connected, coassociative dg coalgebras over R.
- $\Omega : \mathbf{Coalg}_R \to \mathbf{Alg}_R$ is the (reduced) cobar construction.
- $\mathscr{B} : \mathbf{Alg}_R \to \mathbf{Coalg}_R$ is the (reduced) bar construction.

Power maps on Hopf algebras

Let (H, Δ, μ) be a (dg) Hopf algebra, and let $f, g \in \mathrm{End}_R(H)$.

The convolution product $f * g$ of f and g is the composite

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H.$$

The r^{th}-power map on H is the R-linear endomorphism

$$\lambda_r = Id_H^{*r} : H \to H.$$

Philosophy of this lecture: algebra case

When A is an algebra such that $\mathscr{B}A$ is actually a Hopf algebra and therefore admits power maps, construct a lift of the power maps to the Hochschild complex of A.

This lift was known ([BFG], [V]) to be topologically meaningful when $A = \mathcal{A}^*_{PL}(X;\mathbb{Q})$:

> *The algebraic power maps model the topological power maps on $\mathcal{L}X$, the free loop space on X, over $\mathbb{Q}$.*

Philosophy of this lecture: coalgebra case

Similarly, when C is a coalgebra such that ΩC is actually a Hopf algebra, construct a lift of the power maps to the "coHochschild complex" of C.

We show that this lift is topologically meaningful if

- $C = C_* X$, where X is a double suspension, or
- $C = C_*(X; \mathbb{Z}/2\mathbb{Z})$, where $X = \Sigma \bigvee_{j \in J} \mathbb{R}P^{n_j}_{k_j}$:

  The algebraic power maps model the topological power maps on $\mathcal{L}X$, over $\mathbb{Z}$, respectively $\mathbb{Z}/2\mathbb{Z}$.

Why do we care?

- The algebraic power map is essential to the construction of our algebraic model for the spectrum homology of the topological cyclic homology $TC(X; p)$ of a space X.
- Possible purely algebraic applications, e.g., Hodge decompositions?

The classical Hochschild complex

The Hochschild complex is a functor

$$\mathscr{H} : \mathbf{Alg}_R \to \mathbf{Ch}_R$$

such that for all algebras A, there is a twisted tensor extension of chain complexes

$$A \hookrightarrow \mathscr{H}A \twoheadrightarrow \mathscr{B}A.$$

In particular,

$$\mathscr{B}A = (Ts\overline{A}, d_{\mathscr{B}}) \quad \text{and} \quad \mathscr{H}A = (Ts\overline{A} \otimes A, d_{\mathscr{H}}),$$

where $TV := R \oplus V \oplus V^{\otimes 2} \oplus \cdots$.

Multiplicative structure in the commutative case

If A is a commutative algebra, then $\mathscr{B}A$ is naturally a Hopf algebra, with multiplication given by

$$\mathscr{B}A \otimes \mathscr{B}A \xrightarrow{E-Z} \mathscr{B}(A \otimes A) \xrightarrow{\mathscr{B}m_A} \mathscr{B}A.$$

Moreover, the multiplication on $\mathscr{B}A$ lifts to $\mathscr{H}A$, so that

$$A \hookrightarrow \mathscr{H}A \twoheadrightarrow \mathscr{B}A$$

becomes a sequence of dg algebra maps.

Power maps in the commutative case

[Loday,...] If A is a commutative algebra, then the r^{th}-power map

$$\lambda_r : \mathscr{B}A \to \mathscr{B}A$$

lifts to an "r^{th}-power map"

$$\widehat{\lambda}_r : \mathscr{H}A \to \mathscr{H}A$$

such that

$$\begin{array}{ccccc} A & \longrightarrow & \mathscr{H}A & \longrightarrow & \mathscr{B}A \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \widehat{\lambda}_r} & & \downarrow{\scriptstyle \lambda_r} \\ A & \longrightarrow & \mathscr{H}A & \longrightarrow & \mathscr{B}A \end{array}$$

commutes.

Free loop spaces

Let X be a topological space. The free loop space on X

$$\mathcal{L}X := \mathrm{Map}(S^1, X).$$

Evaluation at the basepoint gives a fiber sequence

$$\Omega X \hookrightarrow \mathcal{L}X \twoheadrightarrow X.$$

Power maps on loop spaces

The r^{th}-power maps on ΩX and on $\mathcal{L}X$

$$\ell_r : \Omega X \to \Omega X \quad \text{and} \quad \widehat{\ell}_r : \mathcal{L}X \to \mathcal{L}X$$

are defined by $\widehat{\ell}_r(\alpha)(z) = \alpha(z^r)$ for all $z \in S^1$ and $\widehat{\ell}_r = \ell_r|_{\Omega X}$, so that

$$\begin{array}{ccccc} \Omega X & \longrightarrow & \mathcal{L}X & \longrightarrow & X \\ \downarrow{\scriptstyle \ell_r} & & \downarrow{\scriptstyle \widehat{\ell}_r} & & \downarrow{\scriptstyle =} \\ \Omega X & \longrightarrow & \mathcal{L}X & \longrightarrow & X \end{array}$$

commutes.

Note that

$$\widehat{\ell}_r = \mu^{(r-1)}\Delta^{(r-1)} : \mathcal{L}X \to \mathcal{L}X \underset{X}{\times} \cdots \underset{X}{\times} \mathcal{L}X \to \mathcal{L}X.$$

The power map is given by iterated loop concatenation.

Parallels between algebra and topology

Compare

$$\begin{array}{ccccc}
C^*X & \longrightarrow & C^*\mathcal{L}X & \longrightarrow & C^*\Omega X \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle C^*\widehat{\ell}_r} & & \downarrow{\scriptstyle C^*\ell_r} \\
C^*X & \longrightarrow & C^*\mathcal{L}X & \longrightarrow & C^*\Omega X
\end{array}$$

and

$$\begin{array}{ccccc}
A & \longrightarrow & \mathscr{H}A & \longrightarrow & \mathscr{B}A \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \widehat{\lambda}_r} & & \downarrow{\scriptstyle \lambda_r} \\
A & \longrightarrow & \mathscr{H}A & \longrightarrow & \mathscr{B}A.
\end{array}$$

Topological relevance: the rational case

[BFG, V]
If X is a simplicial complex, then there is a commuting diagram

$$\begin{array}{ccc} \widetilde{HH}_{-*}(\mathcal{A}_{PL}(X)) & \xrightarrow[\cong]{a} & \widetilde{H}^*(\mathcal{L}X;\mathbb{Q}) \\ \downarrow{\scriptstyle H_{-*}\widehat{\lambda}_r} & & \downarrow{\scriptstyle H^*\widehat{\ell}_r} \\ \widetilde{HH}_{-*}(\mathcal{A}_{PL}(X)) & \xrightarrow[\cong]{a} & \widetilde{H}^*(\mathcal{L}X;\mathbb{Q}), \end{array}$$

where

- $\mathcal{A}_{PL}(X)$ is the commutative dg algebra of piecewise-linear $\mathbb{Q}$-valued forms on X;
- $\widetilde{HH}_*(A)$ denotes the reduced homology of $\mathscr{H}A$.

Goals of this lecture

- Generalize the Hochschild complex construction to a larger category.
- Provide conditions under which the generalized Hochschild construction admits power maps.
- Explain the topological relevance of the algebraic power maps.

Twisting cochains

Let $(C, \Delta, d) \in \mathbf{Coalg}_R$ and $(A, m, d) \in \mathbf{Alg}_R$.

A linear map $t : C \to A$ of degree -1 such that

$$dt + td = m(t \otimes t)\Delta$$

is a twisting cochain.

$$\begin{aligned} &\{\text{twisting cochains } t : C \to A\} \\ &\cong \{\text{dg alg maps } \alpha_t : \Omega C \to A\} \\ &\cong \{\text{dg coalg maps } \beta_t : C \to \mathscr{B}A\} \end{aligned}$$

Important examples I

The universal twisting cochain $t_\Omega : C \to \Omega C$:

For every twisting cochain $t : C \to A$, the diagram

$$\begin{array}{ccc} C & \xrightarrow{t_\Omega} & \Omega C \\ & {\scriptstyle t}\searrow & \downarrow{\scriptstyle \alpha_t} \\ & & A \end{array}$$

commutes.

Important examples II

The couniversal twisting cochain $t_{\mathscr{B}} : \mathscr{B}A \to A$:

For every twisting cochain $t : C \to A$, the diagram

$$\begin{array}{ccc} \mathscr{B}A & \xrightarrow{t_{\mathscr{B}}} & A \\ {\scriptstyle \beta_t}\uparrow & \nearrow_{t} & \\ C & & \end{array}$$

commutes.

Important examples III

Let K be a reduced simplicial set, and let C_*K denote the normalized chains on K. Let $\mathrm{G}K$ denote the Kan loop group on K (simplicial analog of the based loop space).

Szczarba's twisting cochain

$$t_K : C_*K \to C_*\mathrm{G}K$$

is natural in K and

$$\alpha_K := \alpha_{t_K} : \Omega C_*K \xrightarrow{\simeq} C_*\mathrm{G}K$$

is a quasi-isomorphism of dg algebras for all K.

The Hochschild complex

Let $t : C \to A$ be a twisting cochain.

The Hochschild complex of t, denoted $\mathscr{H}(t)$, is the chain complex with underlying graded R-module $C \otimes A$ and with differential d_t, defined on $c \otimes a \in C \otimes A$ by

$$\begin{aligned} d_t(c \otimes a) = dc \otimes a &\pm c \otimes da \\ &\pm c_j \otimes t(c^j) \cdot a \\ &\pm c^j \otimes a \cdot t(c_j), \end{aligned}$$

where $\Delta(c) = c_j \otimes c^j$.

There is a twisted extension of chain complexes

$$A \hookrightarrow \mathscr{H}(t) \twoheadrightarrow C.$$

Important examples

- $\mathscr{H}(t_{\mathscr{B}})$ is the usual Hochschild complex $\mathscr{H}A$ on A.
- $\mathscr{H}(t_{\Omega})$ is the coHochschild complex $\widehat{\mathscr{H}}C$ on C [HPS].

Naturality of the Hochschild construction I

Given twisting cochains $t : C \to A$ and $t' : C' \to A'$ and a commutative diagram in $\mathbf{Alg}_R$

$$\begin{array}{ccc} \Omega C & \xrightarrow{\alpha_t} & A \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle g} \\ \Omega C' & \xrightarrow{\alpha_{t'}} & A' \end{array}$$

there is a commuting diagram of chain maps

$$\begin{array}{ccccccc} \Omega C & \xrightarrow{\alpha_t} & A & \longrightarrow & \mathscr{H}(t) & \longrightarrow & C \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathscr{H}(\varphi, g)} & & \downarrow{\scriptstyle f} \\ \Omega C' & \xrightarrow{\alpha_{t'}} & A' & \longrightarrow & \mathscr{H}(t') & \longrightarrow & C,' \end{array}$$

where f is the "linear part" of φ.

Naturality of the Hochschild construction II

Given twisting cochains $t : C \to A$ and $t' : C' \to A'$ and a commutative diagram in $\mathbf{Coalg}_R$

$$\begin{array}{ccc} C & \xrightarrow{\beta_t} & \mathscr{B}A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \gamma} \\ C' & \xrightarrow{\beta_{t'}} & \mathscr{B}A' \end{array}$$

there is a commuting diagram of chain maps

$$\begin{array}{ccccccc} A & \longrightarrow & \mathscr{H}(t) & \longrightarrow & C & \xrightarrow{\beta_t} & \mathscr{B}A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathscr{H}(f,\gamma)} & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \gamma} \\ A' & \longrightarrow & \mathscr{H}(t') & \longrightarrow & C' & \xrightarrow{\beta_{t'}} & \mathscr{B}A' \end{array}$$

where g is the “linear part” of γ.

Special case of naturality I

If $f : C \to C'$ is a coalgebra map and $g : A \to A'$ is an algebra map, then each of the three diagrams below commutes if and only if the other two do.

$$
\begin{array}{ccc}
\Omega C & \xrightarrow{\alpha_t} & A \\
{\scriptstyle \Omega f}\downarrow & & \downarrow{\scriptstyle g} \\
\Omega C' & \xrightarrow{\alpha_{t'}} & A'
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{t} & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
C' & \xrightarrow{t'} & A'
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{\beta_t} & \mathscr{B}A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathscr{B}g} \\
C' & \xrightarrow{\beta_{t'}} & \mathscr{B}A'
\end{array}
$$

Special case of naturality II

Thus, if

$$\begin{array}{ccc} C & \xrightarrow{t} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C' & \xrightarrow{t'} & A' \end{array}$$

commutes, then there is a commuting diagram of chain maps

$$\begin{array}{ccccccccc} \Omega C & \xrightarrow{\alpha_t} & A & \longrightarrow & \mathscr{H}(t) & \longrightarrow & C & \xrightarrow{\beta_t} & \mathscr{B}A \\ {\scriptstyle \Omega f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle \mathscr{H}(f,g)}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathscr{B}g} \\ \Omega C' & \xrightarrow{\alpha_{t'}} & A' & \longrightarrow & \mathscr{H}(t') & \longrightarrow & C' & \xrightarrow{\beta_{t'}} & \mathscr{B}A'. \end{array}$$

Hirsch coalgebras: definition

A Hirsch coalgebra is a dg coalgebra C, together with a dg algebra map

$$\psi : \Omega C \to \Omega C \otimes \Omega C$$

such that $(\Omega C, \psi)$ is a dg Hopf algebra.

A Hirsch coalgebra (C, ψ) is balanced if ψ is cocommutative.

Hirsch coalgebras: examples I

Let C be a dg coalgebra with diagonal

$$\Delta : C \to C \otimes C.$$

If Δ is cocommutative, and therefore a coalgebra map, then $(C, \Omega\Delta)$ is a balanced Hirsch coalgebra.

Hirsch coalgebras: examples II

[Baues, HPST, HPS]

If K is a reduced simplicial set, then there is a natural comultiplication

$$\psi_K : \Omega C_* K \to \Omega C_* K \otimes \Omega C_* K$$

with respect to which

$$\alpha_K : \Omega C_* K \xrightarrow{\simeq} C_* \mathrm{G} K$$

is strongly homotopy comultiplicative.

Moreover, if $C_* K$ is cocommutative (e.g., if K is simplicial suspension), then $(C_* \mathrm{E} K, \psi_{\mathrm{E}K})$ is a balanced Hirsch coalgebra (E $=$ simplicial suspension functor).

Hirsch coalgebras: examples III

[HR]

If H is a dg Hopf algebra, then there is natural comultiplication

$$\psi_H : \Omega\mathscr{B}H \to \Omega\mathscr{B}H \otimes \Omega\mathscr{B}H$$

with respect to which the natural algebra map

$$\varepsilon_H : \Omega\mathscr{B}H \xrightarrow{\simeq} H$$

is a map of Hopf algebras.

The existence theorem: hypotheses

Given

- a Hirsch coalgebra (C, ψ),
- a dg Hopf algebra (H, Δ), and
- a twisting cochain $t : C \to H$

such that

- $\alpha_t : (\Omega C, \psi) \to (H, \Delta)$ is a map of Hopf algebras, and
- $\tau \Delta t = \Delta t : C \to H \otimes H$.

($\tau : H \otimes H \xrightarrow{\cong} H \otimes H$ is the symmetry isomorphism.)

The existence theorem: conclusion

Then, for all $r \geq 1$, there exists a natural chain map

$$\widehat{\lambda}_r : \mathscr{H}(t) \to \mathscr{H}(t)$$

such that

$$\begin{array}{ccccc} H & \longrightarrow & \mathscr{H}(t) & \longrightarrow & C \\ \downarrow{\scriptstyle \lambda_r} & & \downarrow{\scriptstyle \widehat{\lambda}_r} & & \downarrow{\scriptstyle =} \\ H & \longrightarrow & \mathscr{H}(t) & \longrightarrow & C \end{array}$$

commutes.

In particular, if $(\Omega C, \psi)$ is primitively generated, then $\widehat{\lambda}_r = Id_C \otimes \lambda_r$.

Special case: balanced Hirsch coalgebras

If (C, ψ) is a balanced Hirsch coalgebra, then for all $r \geq 1$, there exists a natural chain map

$$\widehat{\lambda}_r : \widehat{\mathscr{H}}C \to \widehat{\mathscr{H}}C$$

such that

$$\begin{array}{ccccc} \Omega C & \longrightarrow & \widehat{\mathscr{H}}C & \longrightarrow & C \\ \downarrow{\scriptstyle \lambda_r} & & \downarrow{\scriptstyle \widehat{\lambda}_r} & & \downarrow{\scriptstyle =} \\ \Omega C & \longrightarrow & \widehat{\mathscr{H}}C & \longrightarrow & C \end{array}$$

commutes.

Special case: cocommutative Hopf algebras

If (H, Δ) is a cocommutative Hopf algebra, then for all $r \geq 1$, there exists

$$\widehat{\lambda}_r : \mathscr{H} H \to \mathscr{H} H$$

such that

$$\begin{array}{ccccc} H & \longrightarrow & \mathscr{H} H & \longrightarrow & \mathscr{B} H \\ \downarrow{\scriptstyle \lambda_r} & & \downarrow{\scriptstyle \widehat{\lambda}_r} & & \downarrow{\scriptstyle =} \\ H & \longrightarrow & \mathscr{H} H & \longrightarrow & \mathscr{B} H \end{array}$$

commutes.

A small, explicit model for free loop spaces

[HPS]

If X is a topological space such that $X \simeq |K|$ for some reduced simplicial set K, then there is a quasi-isomorphism

$$\zeta : \widehat{\mathscr{H}} C_* K \xrightarrow{\simeq} S_* \mathcal{L} X.$$

Here, S_* denotes integral singular chains.

Compatability with power maps: integral case

[HR]

If $K = \mathrm{E}^2 L$, where L is any simplicial set, and $X \simeq |K|$, then for all $r \geq 1$,

$$\begin{array}{ccc} \widehat{\mathscr{H}} C_* K & \xrightarrow[\simeq]{\zeta} & S_* \mathcal{L} X \\ \downarrow{\scriptstyle \widehat{\lambda}_r} & & \downarrow{\scriptstyle S_* \widehat{\ell}_r} \\ \widehat{\mathscr{H}} C_* K & \xrightarrow[\simeq]{\zeta} & S_* \mathcal{L} X \end{array}$$

commutes up to chain homotopy.

Remark

The condition $K = \mathrm{E}^2 L$ implies that $(C_* K, \psi_K)$ is a balanced Hirsch coalgebra and therefore that the power map $\widehat{\lambda}_r$ exists.

Compatability with power maps: mod 2 case

[HR] Let $X = \Sigma \bigvee_{j \in J} \mathbb{R}P^{n_j}_{k_j}$.

There exists a finite-type simplicial set K such that for all $r \geq 1$,

$$\begin{array}{ccc}
\widehat{\mathscr{H}}C_*(K;\mathbb{F}_2) & \xrightarrow[\simeq]{\zeta} & S_*(\mathcal{L}X;\mathbb{F}_2) \\
\Big\downarrow \widehat{\lambda}_r & & \Big\downarrow S_*\widehat{\ell}_r \\
\widehat{\mathscr{H}}C_*(K;\mathbb{F}_2) & \xrightarrow[\simeq]{\zeta} & S_*(\mathcal{L}X;\mathbb{F}_2)
\end{array}$$

commutes up to chain homotopy.

Remark

Any such X admits a simplicial model K such that $\big(C_*(K;\mathbb{F}_2), \psi_K\big)$ is a balanced Hirsch coalgebra, which implies that the power map $\widehat{\lambda}_r$ exists.

Sketch of the proof of compatibility I

[BHM] Let G be a topological group, $\mathrm{Z}G$ its cyclic nerve and $\mathrm{B}G$ its usual nerve.

Then there exists a homotopy equivalence

$$h : |\mathrm{Z}G| \xrightarrow{\simeq} \mathcal{L}|\mathrm{B}G|$$

and an operator

$$\ell_r^{cyc} : |\mathrm{Z}G| \to |\mathrm{Z}G|$$

such that

$$\begin{array}{ccc} |\mathrm{Z}G| & \xrightarrow[\simeq]{h} & \mathcal{L}|\mathrm{B}G| \\ \ell_r^{cyc}\downarrow & & \downarrow\widehat{\ell}_r \\ |\mathrm{Z}G| & \xrightarrow[\simeq]{h} & \mathcal{L}|\mathrm{B}G| \end{array}$$

commutes.

Sketch of the proof of compatibility II

Let G be a topological group, $\mathrm{Z}G$ its cyclic nerve and $\mathrm{B}G$ its usual nerve.

There is a simplicial map $\ell_r^{simp} : \mathrm{Z}G \to \mathrm{Z}G$ such that

$$|\ell_r^{simp}| \simeq \ell_r^{cyc} : |\mathrm{Z}G| \to |\mathrm{Z}G|.$$

Sketch of the proof of compatibility III

- For any simplicial group G, there is a twisted cartesian product (Kan fibration)

$$G \hookrightarrow \mathrm{H}G \twoheadrightarrow \mathrm{W}G,$$

where W is Kan's classifying space functor and HG is the simplicial Hochschild construction, admitting a power map λ_r^{hoch}.

- For any reduced simplicial set K, there is a twisted cartesian product (Kan fibration)

$$\mathrm{G}K \hookrightarrow \widehat{\mathrm{H}}K \twoheadrightarrow K,$$

where $\widehat{\mathrm{H}}G$ is the simplicial coHochschild construction, admitting a power map λ_r^{cohoch}.

Sketch of the proof of compatibility IV

Let K be a reduced simplicial set, and let $G = |\mathrm{G}K|$. Then there is a commuting diagram

$$
\begin{array}{ccccc}
C_*\mathrm{G}K & \longrightarrow & C_*\widehat{\mathrm{H}}K & \twoheadrightarrow & C_*K \\
\| & & \downarrow \simeq & & \downarrow \simeq \\
C_*\mathrm{G}K & \longrightarrow & C_*\mathrm{HG}K & \twoheadrightarrow & C_*\mathrm{WG}K \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
S_*|\mathrm{C}G| & \longrightarrow & S_*|\mathrm{Z}G| & \twoheadrightarrow & S_*|\mathrm{B}G| \\
\downarrow \simeq & & \downarrow \simeq & & \| \\
S_*G & \longrightarrow & S_*\mathcal{L}|\mathrm{B}G| & \twoheadrightarrow & S_*|\mathrm{B}G|,
\end{array}
$$

compatible at every stage with power maps.

Sketch of the proof of compatibility V

(With contribution from [HPS])

If $K = \mathrm{E}^2 L$, where L is any simplicial set, then there is a natural quasi-isomorphism

$$\theta_K : \widehat{\mathscr{H}} C_* K \xrightarrow{\simeq} C_* \widehat{\mathrm{H}} K$$

such that

$$\begin{array}{ccc} \widehat{\mathscr{H}} C_* K & \xrightarrow{\theta_K} & C_* \widehat{\mathrm{H}} K \\ \downarrow{\scriptstyle \widehat{\lambda}_r} & & \downarrow{\scriptstyle C_* \lambda_r^{cohoch}} \\ \widehat{\mathscr{H}} C_* K & \xrightarrow{\theta_K} & C_* \widehat{\mathrm{H}} K \end{array}$$

commutes up to chain homotopy.