

Power Maps in Algebra and Topology

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Preface

The case of
commutative
algebras

The Hochschild
complex of a
twisting cochain

Power maps on
the Hochschild
complex

Topological
relevance

Joint work with John Rognes.

Outline

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- 3 The Hochschild complex of a twisting cochain
- 4 Power maps on the Hochschild complex
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- R is a PID
- \mathbf{Ch}_R is the category of chain complexes of R -modules that are R -free as graded R -modules.
- \mathbf{Alg}_R is the category of augmented, associative dg algebras over R .
- \mathbf{Coalg}_R is the category of coaugmented, connected, coassociative dg coalgebras over R .
- $\Omega : \mathbf{Coalg}_R \rightarrow \mathbf{Alg}_R$ is the (reduced) cobar construction.
- $\mathcal{B} : \mathbf{Alg}_R \rightarrow \mathbf{Coalg}_R$ is the (reduced) bar construction.

Power maps on Hopf algebras

Let (H, Δ, μ) be a (dg) Hopf algebra, and let $f, g \in \text{End}_R(H)$.

The **convolution product** $f * g$ of f and g is the composite

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H.$$

The **r^{th} -power map** on H is the R -linear endomorphism

$$\lambda_r = \text{Id}_H^{*r} : H \rightarrow H.$$

Philosophy of this lecture: algebra case

When A is an algebra such that $\mathcal{B}A$ is actually a **Hopf algebra** and therefore admits power maps, construct a lift of the power maps to the Hochschild complex of A .

This lift was known ([BFG], [V]) to be topologically meaningful when $A = \mathcal{A}_{PL}^*(X; \mathbb{Q})$:

The algebraic power maps model the topological power maps on $\mathcal{L}X$, the free loop space on X , over \mathbb{Q} .

Philosophy of this lecture: coalgebra case

Similarly, when C is a coalgebra such that ΩC is actually a **Hopf algebra**, construct a lift of the power maps to the “coHochschild complex” of C .

We show that this lift is topologically meaningful if

- $C = C_*X$, where X is a double suspension, or
- $C = C_*(X; \mathbb{Z}/2\mathbb{Z})$, where $X = \Sigma \bigvee_{j \in J} \mathbb{R}P_{k_j}^{n_j}$:

The algebraic power maps model the topological power maps on $\mathcal{L}X$, over \mathbb{Z} , respectively $\mathbb{Z}/2\mathbb{Z}$.

Why do we care?

- The algebraic power map is essential to the construction of our algebraic model for the spectrum homology of the topological cyclic homology $TC(X; p)$ of a space X .
- Possible purely algebraic applications, e.g., Hodge decompositions?

The classical Hochschild complex

The **Hochschild complex** is a functor

$$\mathcal{H} : \mathbf{Alg}_R \rightarrow \mathbf{Ch}_R$$

such that for all algebras A , there is a twisted tensor extension of chain complexes

$$A \hookrightarrow \mathcal{H}A \twoheadrightarrow \mathcal{B}A.$$

In particular,

$$\mathcal{B}A = (Ts\bar{A}, d_{\mathcal{B}}) \quad \text{and} \quad \mathcal{H}A = (Ts\bar{A} \otimes A, d_{\mathcal{H}}),$$

where $TV := R \oplus V \oplus V^{\otimes 2} \oplus \dots$.

Multiplicative structure in the commutative case

If A is a **commutative** algebra, then $\mathcal{B}A$ is naturally a **Hopf algebra**, with multiplication given by

$$\mathcal{B}A \otimes \mathcal{B}A \xrightarrow{E-Z} \mathcal{B}(A \otimes A) \xrightarrow{\mathcal{B}m_A} \mathcal{B}A.$$

Moreover, the multiplication on $\mathcal{B}A$ lifts to $\mathcal{H}A$, so that

$$A \hookrightarrow \mathcal{H}A \twoheadrightarrow \mathcal{B}A$$

becomes a sequence of dg algebra maps.

Power maps in the commutative case

[Loday,...] If A is a commutative algebra, then the r^{th} -power map

$$\lambda_r : BA \rightarrow BA$$

lifts to an “ r^{th} -power map”

$$\widehat{\lambda}_r : \mathcal{H}A \rightarrow \mathcal{H}A$$

such that

$$\begin{array}{ccccc} A & \longrightarrow & \mathcal{H}A & \longrightarrow & BA \\ \cong \downarrow & & \widehat{\lambda}_r \downarrow & & \lambda_r \downarrow \\ A & \longrightarrow & \mathcal{H}A & \longrightarrow & BA \end{array}$$

commutes.

Free loop spaces

Let X be a topological space. The **free loop space** on X

$$\mathcal{L}X := \text{Map}(S^1, X).$$

Evaluation at the basepoint gives a fiber sequence

$$\Omega X \hookrightarrow \mathcal{L}X \twoheadrightarrow X.$$

Power maps on loop spaces

The r^{th} -power maps on ΩX and on $\mathcal{L}X$

$$\ell_r : \Omega X \rightarrow \Omega X \quad \text{and} \quad \widehat{\ell}_r : \mathcal{L}X \rightarrow \mathcal{L}X$$

are defined by $\widehat{\ell}_r(\alpha)(z) = \alpha(z^r)$ for all $z \in S^1$ and $\widehat{\ell}_r = \ell_r|_{\Omega X}$, so that

$$\begin{array}{ccccc} \Omega X & \longrightarrow & \mathcal{L}X & \longrightarrow & X \\ \ell_r \downarrow & & \widehat{\ell}_r \downarrow & & = \downarrow \\ \Omega X & \longrightarrow & \mathcal{L}X & \longrightarrow & X \end{array}$$

commutes.

Note that

$$\widehat{\ell}_r = \mu^{(r-1)} \Delta^{(r-1)} : \mathcal{L}X \rightarrow \mathcal{L}X \underset{X}{\times} \cdots \underset{X}{\times} \mathcal{L}X \rightarrow \mathcal{L}X.$$

The power map is given by **iterated loop concatenation**.

Parallels between algebra and topology

Compare

$$\begin{array}{ccccc} C^*X & \longrightarrow & C^*\mathcal{L}X & \longrightarrow & C^*\Omega X \\ \downarrow = & & \downarrow C^*\widehat{\ell}_r & & \downarrow C^*l_r \\ C^*X & \longrightarrow & C^*\mathcal{L}X & \longrightarrow & C^*\Omega X \end{array}$$

and

$$\begin{array}{ccccc} A & \longrightarrow & \mathcal{H}A & \longrightarrow & BA \\ \downarrow = & & \downarrow \widehat{\lambda}_r & & \downarrow \lambda_r \\ A & \longrightarrow & \mathcal{H}A & \longrightarrow & BA. \end{array}$$

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Topological relevance: the rational case

[BFG, V]

If X is a simplicial complex, then there is a commuting diagram

$$\begin{array}{ccc} \widetilde{HH}_{-*}(\mathcal{A}_{PL}(X)) & \xrightarrow{\cong} & \widetilde{H}^*(\mathcal{L}X; \mathbb{Q}) \\ \downarrow H_{-*}\widehat{\lambda}_r & & \downarrow H^*\widehat{\ell}_r \\ \widetilde{HH}_{-*}(\mathcal{A}_{PL}(X)) & \xrightarrow{\cong} & \widetilde{H}^*(\mathcal{L}X; \mathbb{Q}), \end{array}$$

where

- $\mathcal{A}_{PL}(X)$ is the commutative dg algebra of piecewise-linear \mathbb{Q} -valued forms on X ;
- $\widetilde{HH}_*(A)$ denotes the reduced homology of $\mathcal{H}A$.

Goals of this lecture

- Generalize the Hochschild complex construction to a larger category.
- Provide conditions under which the generalized Hochschild construction admits power maps.
- Explain the topological relevance of the algebraic power maps.

Twisting cochains

Let $(C, \Delta, d) \in \mathbf{Coalg}_R$ and $(A, m, d) \in \mathbf{Alg}_R$.

A linear map $t : C \rightarrow A$ of degree -1 such that

$$dt + td = m(t \otimes t)\Delta$$

is a **twisting cochain**.

$$\begin{aligned} & \{\text{twisting cochains } t : C \rightarrow A\} \\ & \cong \{\text{dg alg maps } \alpha_t : \Omega C \rightarrow A\} \\ & \cong \{\text{dg coalg maps } \beta_t : C \rightarrow \mathcal{B}A\} \end{aligned}$$

Important examples I

The **universal twisting cochain** $t_\Omega : C \rightarrow \Omega C$:

For every twisting cochain $t : C \rightarrow A$, the diagram

$$\begin{array}{ccc} C & \xrightarrow{t_\Omega} & \Omega C \\ & \searrow t & \downarrow \alpha_t \\ & & A \end{array}$$

commutes.

Important examples II

The **couniversal twisting cochain** $t_{\mathcal{B}} : \mathcal{B}A \rightarrow A$:

For every twisting cochain $t : C \rightarrow A$, the diagram

$$\begin{array}{ccc} \mathcal{B}A & \xrightarrow{t_{\mathcal{B}}} & A \\ \beta_t \uparrow & \nearrow t & \\ C & & \end{array}$$

commutes.

Important examples III

Let K be a reduced simplicial set, and let C_*K denote the normalized chains on K . Let GK denote the Kan loop group on K (simplicial analog of the based loop space).

Szczarba's twisting cochain

$$t_K : C_*K \rightarrow C_*GK$$

is natural in K and

$$\alpha_K := \alpha_{t_K} : \Omega C_*K \xrightarrow{\cong} C_*GK$$

is a quasi-isomorphism of dg algebras for all K .

The Hochschild complex

Let $t : C \rightarrow A$ be a twisting cochain.

The **Hochschild complex** of t , denoted $\mathcal{H}(t)$, is the chain complex with underlying graded R -module $C \otimes A$ and with differential d_t , defined on $c \otimes a \in C \otimes A$ by

$$\begin{aligned}d_t(c \otimes a) &= dc \otimes a \pm c \otimes da \\ &\quad \pm c_j \otimes t(c^j) \cdot a \\ &\quad \pm c^j \otimes a \cdot t(c_j),\end{aligned}$$

where $\Delta(c) = c_j \otimes c^j$.

There is a twisted extension of chain complexes

$$A \hookrightarrow \mathcal{H}(t) \twoheadrightarrow C.$$

Important examples

- $\mathcal{H}(t_{\mathcal{B}})$ is the usual Hochschild complex $\mathcal{H}A$ on A .
- $\mathcal{H}(t_{\Omega})$ is the coHochschild complex $\widehat{\mathcal{H}}C$ on C [HPS].

Naturality of the Hochschild construction I

Given twisting cochains $t : C \rightarrow A$ and $t' : C' \rightarrow A'$ and a commutative diagram in \mathbf{Alg}_R

$$\begin{array}{ccc} \Omega C & \xrightarrow{\alpha_t} & A \\ \varphi \downarrow & & \downarrow g \\ \Omega C' & \xrightarrow{\alpha_{t'}} & A' \end{array}$$

there is a commuting diagram of chain maps

$$\begin{array}{ccccccc} \Omega C & \xrightarrow{\alpha_t} & A & \longrightarrow & \mathcal{H}(t) & \longrightarrow & C \\ \varphi \downarrow & & \downarrow g & & \downarrow \mathcal{H}(\varphi, g) & & \downarrow f \\ \Omega C' & \xrightarrow{\alpha_{t'}} & A' & \longrightarrow & \mathcal{H}(t') & \longrightarrow & C' \end{array}$$

where f is the “linear part” of φ .

Naturality of the Hochschild construction II

Given twisting cochains $t : C \rightarrow A$ and $t' : C' \rightarrow A'$ and a commutative diagram in \mathbf{Coalg}_R

$$\begin{array}{ccc} C & \xrightarrow{\beta_t} & \mathcal{B}A \\ f \downarrow & & \downarrow \gamma \\ C' & \xrightarrow{\beta_{t'}} & \mathcal{B}A' \end{array}$$

there is a commuting diagram of chain maps

$$\begin{array}{ccccccc} A & \longrightarrow & \mathcal{H}(t) & \longrightarrow & C & \xrightarrow{\beta_t} & \mathcal{B}A \\ g \downarrow & & \mathcal{H}(f, \gamma) \downarrow & & f \downarrow & & \downarrow \gamma \\ A' & \longrightarrow & \mathcal{H}(t') & \longrightarrow & C' & \xrightarrow{\beta_{t'}} & \mathcal{B}A' \end{array}$$

where g is the “linear part” of γ .

Special case of naturality I

If $f : C \rightarrow C'$ is a coalgebra map and $g : A \rightarrow A'$ is an algebra map, then each of the three diagrams below commutes if and only if the other two do.

$$\begin{array}{ccc} \Omega C & \xrightarrow{\alpha_t} & A \\ \Omega f \downarrow & & \downarrow g \\ \Omega C' & \xrightarrow{\alpha_{t'}} & A' \end{array}$$

$$\begin{array}{ccc} C & \xrightarrow{t} & A \\ f \downarrow & & \downarrow g \\ C' & \xrightarrow{t'} & A' \end{array}$$

$$\begin{array}{ccc} C & \xrightarrow{\beta_t} & \mathcal{B}A \\ f \downarrow & & \downarrow \mathcal{B}g \\ C' & \xrightarrow{\beta_{t'}} & \mathcal{B}A' \end{array}$$

Special case of naturality II

Thus, if

$$\begin{array}{ccc}
 C & \xrightarrow{t} & A \\
 f \downarrow & & \downarrow g \\
 C' & \xrightarrow{t'} & A'
 \end{array}$$

commutes, then there is a commuting diagram of chain maps

$$\begin{array}{ccccccccc}
 \Omega C & \xrightarrow{\alpha_t} & A & \longrightarrow & \mathcal{H}(t) & \longrightarrow & C & \xrightarrow{\beta_t} & \mathcal{B}A \\
 \Omega f \downarrow & & g \downarrow & & \mathcal{H}(f,g) \downarrow & & f \downarrow & & \mathcal{B}g \downarrow \\
 \Omega C' & \xrightarrow{\alpha_{t'}} & A' & \longrightarrow & \mathcal{H}(t') & \longrightarrow & C' & \xrightarrow{\beta_{t'}} & \mathcal{B}A'.
 \end{array}$$

Hirsch coalgebras: definition

A **Hirsch coalgebra** is a dg coalgebra C , together with a dg algebra map

$$\psi : \Omega C \rightarrow \Omega C \otimes \Omega C$$

such that $(\Omega C, \psi)$ is a dg Hopf algebra.

A Hirsch coalgebra (C, ψ) is **balanced** if ψ is cocommutative.

Hirsch coalgebras: examples I

Let C be a dg coalgebra with diagonal

$$\Delta : C \rightarrow C \otimes C.$$

If Δ is **cocommutative**, and therefore a coalgebra map, then $(C, \Omega\Delta)$ is a balanced Hirsch coalgebra.

Hirsch coalgebras: examples II

[Baues, HPST, HPS]

If K is a reduced simplicial set, then there is a natural comultiplication

$$\psi_K : \Omega C_* K \rightarrow \Omega C_* K \otimes \Omega C_* K$$

with respect to which

$$\alpha_K : \Omega C_* K \xrightarrow{\cong} C_* GK$$

is strongly homotopy comultiplicative.

Moreover, if $C_* K$ is cocommutative (e.g., if K is simplicial suspension), then $(C_* EK, \psi_{EK})$ is a **balanced** Hirsch coalgebra ($E =$ simplicial suspension functor).

Hirsch coalgebras: examples III

[HR]

If H is a dg Hopf algebra, then there is natural comultiplication

$$\psi_H : \Omega \mathcal{B}H \rightarrow \Omega \mathcal{B}H \otimes \Omega \mathcal{B}H$$

with respect to which the natural algebra map

$$\varepsilon_H : \Omega \mathcal{B}H \xrightarrow{\cong} H$$

is a map of Hopf algebras.

The existence theorem: hypotheses

Given

- a Hirsch coalgebra (C, ψ) ,
- a dg Hopf algebra (H, Δ) , and
- a twisting cochain $t : C \rightarrow H$

such that

- $\alpha_t : (\Omega C, \psi) \rightarrow (H, \Delta)$ is a map of Hopf algebras, and
- $\tau \Delta t = \Delta t : C \rightarrow H \otimes H$.

($\tau : H \otimes H \xrightarrow{\cong} H \otimes H$ is the symmetry isomorphism.)

The existence theorem: conclusion

Then, for all $r \geq 1$, there exists a natural chain map

$$\widehat{\lambda}_r : \mathcal{H}(t) \rightarrow \mathcal{H}(t)$$

such that

$$\begin{array}{ccccc} H & \longrightarrow & \mathcal{H}(t) & \longrightarrow & C \\ \lambda_r \downarrow & & \widehat{\lambda}_r \downarrow & & = \downarrow \\ H & \longrightarrow & \mathcal{H}(t) & \longrightarrow & C \end{array}$$

commutes.

In particular, if $(\Omega C, \psi)$ is primitively generated, then $\widehat{\lambda}_r = Id_C \otimes \lambda_r$.

Special case: balanced Hirsch coalgebras

If (C, ψ) is a balanced Hirsch coalgebra, then for all $r \geq 1$, there exists a natural chain map

$$\widehat{\lambda}_r : \widehat{\mathcal{H}}C \rightarrow \widehat{\mathcal{H}}C$$

such that

$$\begin{array}{ccccc} \Omega C & \longrightarrow & \widehat{\mathcal{H}}C & \longrightarrow & C \\ \lambda_r \downarrow & & \widehat{\lambda}_r \downarrow & & = \downarrow \\ \Omega C & \longrightarrow & \widehat{\mathcal{H}}C & \longrightarrow & C \end{array}$$

commutes.

Special case: cocommutative Hopf algebras

If (H, Δ) is a cocommutative Hopf algebra, then for all $r \geq 1$, there exists

$$\widehat{\lambda}_r : \mathcal{H}H \rightarrow \mathcal{H}H$$

such that

$$\begin{array}{ccccc} H & \longrightarrow & \mathcal{H}H & \longrightarrow & \mathcal{B}H \\ \lambda_r \downarrow & & \widehat{\lambda}_r \downarrow & & = \downarrow \\ H & \longrightarrow & \mathcal{H}H & \longrightarrow & \mathcal{B}H \end{array}$$

commutes.

A small, explicit model for free loop spaces

[HPS]

If X is a topological space such that $X \simeq |K|$ for some reduced simplicial set K , then there is a quasi-isomorphism

$$\zeta : \widehat{\mathcal{H}C}_*K \xrightarrow{\simeq} S_*\mathcal{L}X.$$

Here, S_* denotes integral singular chains.

Compatibility with power maps: integral case

[HR]

If $K = E^2L$, where L is any simplicial set, and $X \simeq |K|$, then for all $r \geq 1$,

$$\begin{array}{ccc} \widehat{\mathcal{H}}C_*K & \xrightarrow[\simeq]{\zeta} & S_*\mathcal{L}X \\ \widehat{\lambda}_r \downarrow & & S_*\widehat{\ell}_r \downarrow \\ \widehat{\mathcal{H}}C_*K & \xrightarrow[\simeq]{\zeta} & S_*\mathcal{L}X \end{array}$$

commutes up to chain homotopy.

Remark

The condition $K = E^2L$ implies that (C_*K, ψ_K) is a balanced Hirsch coalgebra and therefore that the power map $\widehat{\lambda}_r$ exists.

Compatibility with power maps: mod 2 case

[HR] Let $X = \Sigma \bigvee_{j \in J} \mathbb{R}P_{k_j}^{n_j}$.

There exists a finite-type simplicial set K such that for all $r \geq 1$,

$$\begin{array}{ccc} \widehat{\mathcal{H}}\mathcal{C}_*(K; \mathbb{F}_2) & \xrightarrow[\simeq]{\zeta} & \mathcal{S}_*(\mathcal{L}X; \mathbb{F}_2) \\ \widehat{\lambda}_r \downarrow & & \mathcal{S}_* \widehat{\ell}_r \downarrow \\ \widehat{\mathcal{H}}\mathcal{C}_*(K; \mathbb{F}_2) & \xrightarrow[\simeq]{\zeta} & \mathcal{S}_*(\mathcal{L}X; \mathbb{F}_2) \end{array}$$

commutes up to chain homotopy.

Remark

Any such X admits a simplicial model K such that $(\mathcal{C}_*(K; \mathbb{F}_2), \psi_K)$ is a balanced Hirsch coalgebra, which implies that the power map $\widehat{\lambda}_r$ exists.

Sketch of the proof of compatibility I

[BHM] Let G be a topological group, ZG its cyclic nerve and BG its usual nerve.

Then there exists a homotopy equivalence

$$h : |ZG| \xrightarrow{\simeq} \mathcal{L}|BG|$$

and an operator

$$\ell_r^{\text{cyc}} : |ZG| \rightarrow |ZG|$$

such that

$$\begin{array}{ccc} |ZG| & \xrightarrow[\simeq]{h} & \mathcal{L}|BG| \\ \ell_r^{\text{cyc}} \downarrow & & \widehat{\ell}_r \downarrow \\ |ZG| & \xrightarrow[\simeq]{h} & \mathcal{L}|BG| \end{array}$$

commutes.

Sketch of the proof of compatibility II

Let G be a topological group, ZG its cyclic nerve and BG its usual nerve.

There is a simplicial map $\ell_r^{simp} : ZG \rightarrow ZG$ such that

$$|\ell_r^{simp}| \simeq \ell_r^{cyc} : |ZG| \rightarrow |ZG|.$$

Sketch of the proof of compatibility III

- For any simplicial group G , there is a twisted cartesian product (Kan fibration)

$$G \hookrightarrow HG \twoheadrightarrow WG,$$

where W is Kan's classifying space functor and HG is the **simplicial Hochschild construction**, admitting a power map λ_r^{hoch} .

- For any reduced simplicial set K , there is a twisted cartesian product (Kan fibration)

$$GK \hookrightarrow \widehat{HK} \twoheadrightarrow K,$$

where \widehat{HG} is the **simplicial coHochschild construction**, admitting a power map λ_r^{cohoch} .

Sketch of the proof of compatibility IV

Let K be a reduced simplicial set, and let $G = |GK|$. Then there is a commuting diagram

$$\begin{array}{ccccc}
 C_* GK & \longrightarrow & C_* \widehat{HK} & \twoheadrightarrow & C_* K \\
 \parallel & & \downarrow \simeq & & \downarrow \simeq \\
 C_* GK & \longrightarrow & C_* HGK & \twoheadrightarrow & C_* WGK \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 S_* |CG| & \longrightarrow & S_* |ZG| & \twoheadrightarrow & S_* |BG| \\
 \downarrow \simeq & & \downarrow \simeq & & \parallel \\
 S_* G & \longrightarrow & S_* \mathcal{L} |BG| & \twoheadrightarrow & S_* |BG|,
 \end{array}$$

compatible at every stage with power maps.

Sketch of the proof of compatibility V

(With contribution from [HPS])

If $K = E^2L$, where L is any simplicial set, then there is a natural quasi-isomorphism

$$\theta_K : \widehat{\mathcal{H}}C_*K \xrightarrow{\cong} C_*\widehat{H}K$$

such that

$$\begin{array}{ccc} \widehat{\mathcal{H}}C_*K & \xrightarrow{\theta_K} & C_*\widehat{H}K \\ \widehat{\lambda}_r \downarrow & & \downarrow C_*\lambda_r^{\text{cohoch}} \\ \widehat{\mathcal{H}}C_*K & \xrightarrow{\theta_K} & C_*\widehat{H}K \end{array}$$

commutes up to chain homotopy.