Homotopic Hopf-Galois Extensions

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Outline

1. History and motivation
2. Co-rings and their comodules
3. Homotopic Hopf-Galois extensions
4. Descent

Hopf-Galois extensions of rings: Data

- Commutative ring $\mathbb{k}$
- Homomorphism of augmented $\mathbb{k}$-algebras $\varphi : B \to A$
- $\mathbb{k}$-bialgebra H, seen as a B-algebra with trivial B-action
- Coassociative, counital morphism $\rho : A \to A \otimes_{\mathbb{k}} H$ of B-algebras

Hopf-Galois extensions of rings: Maps

- The Galois map β_φ:

$$A \underset{B}{\otimes} A \xrightarrow{A \underset{B}{\otimes} \rho} A \underset{B}{\otimes} A \underset{\Bbbk}{\otimes} H \xrightarrow{\mu \otimes H} A \underset{\Bbbk}{\otimes} H.$$

- The corestriction map i_φ:

$$B \to A^{co\,H} := A \underset{H}{\square} \Bbbk = \{a \in A \mid \rho(a) = a \otimes 1\}$$

Hopf-Galois extensions of rings: Definition

The homomorphism $\varphi : B \to A$ is an *H-Hopf-Galois extension* if the Galois map

$$\beta_\varphi : A \underset{B}{\otimes} A \to A \underset{\Bbbk}{\otimes} H$$

and the corestriction map

$$i_\varphi : B \to A^{co\, H}$$

are both isomorphisms.

Hopf-Galois extensions of rings: Examples

1. Let G be a group. Any G-Galois extension $\varphi : B \to A$ is a $\mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z})$-Hopf-Galois extension.
2. If H is a Hopf algebra that is flat as a $\Bbbk$-module and A is a flat $\Bbbk$-algebra, then

$$A \to A \otimes H : a \mapsto a \otimes 1$$

is an H-Hopf-Galois extension.

Generalization to ring spectra

[Rognes]

The unit map $\eta : S \to MU$ is an $S[BU]$-Hopf-Galois extension in a homotopical sense (i.e., replacing isomorphisms by weak equivalences), where

- the diagonal $\Delta : BU \to BU \times BU$ induces the comultiplication $S[BU] \to S[BU] \wedge S[BU]$;
- the Thom diagonal $MU \to MU \wedge BU_+$ gives rise to the coaction of $S[BU]$ on MU; and
- $\beta_\eta : MU \wedge MU \xrightarrow{\simeq} MU \wedge S[BU]$ is the Thom equivalence.

Goals

Categorify the ring-level definition à la Morita, in a homotopical sense:

isomorphisms of objects

$$\downarrow$$

Quillen equivalences of model categories.

Characterize homotopic Hopf-Galois extensions in well-known monoidal model categories.

Determine the role of homotopic Hopf-Galois extensions in descent theory.

The framework

Let $\mathbf{M}$ be a category endowed with

- a monoidal structure: $-\otimes - : \mathbf{M}\times\mathbf{M}\to\mathbf{M}$ and $I\in \operatorname{Ob}\mathbf{M}$ such that

  $$(A\otimes B)\otimes C\cong A\otimes(B\otimes C)\quad\text{and}\quad A\otimes I\cong A\cong I\otimes A;$$

- a model structure: a framework for defining homotopy relations on morphisms, involving distinguished classes of morphisms–weak equivalences, fibrations and cofibrations–satisfying axioms analogous to the properties of the continuous maps with the same names.

Definition

Let A be a monoid in **M**.

An *A-co-ring* is a comonoid in $({}_A\mathbf{Mod}_A, - \underset{A}{\otimes} -)$, i.e., an A-bimodule W endowed with a coassociative, counital comultiplication

$$W \to W \underset{A}{\otimes} W$$

that is a morphism of A-bimodules.

Example

If $A = I$, then ${}_A\mathbf{Mod}_A = \mathbf{M}$, and an A-co-ring is just a comonoid in **M**.

Example

A itself is an A-co-ring, endowed with the comultiplication $A \xrightarrow{\cong} A \underset{A}{\otimes} A$.

Comodules over co-rings

If W is an A-co-ring, then $\mathbf{M}_A^W$ is the category of W-comodules in the category of right A-modules.

An object of $\mathbf{M}_A^W$ is thus a right A-module M together with a coassociative, counital morphism of right A-modules

$$\theta : M \to M \underset{A}{\otimes} W.$$

Example

If $A = I$ and C is a comonoid, then $\mathbf{M}_I^C = \mathbf{Comod}_C$, the category of right C-comodules.

Example

$\mathbf{M}_A^A \cong \mathbf{Mod}_A$.

Basic functors

Let $\gamma : W \to W'$ be a morphism of A-co-rings.

- The forgetful functor $U_W : \mathbf{M}_A^W \to \mathbf{Mod}_A$
- The cofree functor $- \underset{A}{\otimes} W : \mathbf{Mod}_A \to \mathbf{M}_A^W$
- The induced functor $\gamma_* : \mathbf{M}_A^W \to \mathbf{M}_A^{W'}$

Remark

The functor U_W is the left adjoint to $- \underset{A}{\otimes} W$, and γ_* is a left adjoint if $\mathbf{M}_A^W$ admits equalizers.

The trivial/coinvariants adjunction

Let W be an A-co-ring, endowed with a coaugmentation, i.e., a morphism of A-co-rings $\eta : A \to W$.

The trivial W-comodule functor is

$$\mathrm{Triv}_W = \eta_* : \mathbf{Mod}_A \to \mathbf{M}_A^W.$$

If $\mathbf{Mod}_A$ admits equalizers, then the W-coinvariants functor

$$\mathrm{Coinv}_W : \mathbf{M}_A^W \to \mathbf{Mod}_A$$

is the right adjoint to Triv_W, defined by

$$\mathrm{Coinv}(M, \theta) = M^{co\,W} = \mathrm{equal}(M \underset{\theta}{\overset{M \underset{A}{\otimes} \eta}{\rightrightarrows}} M \underset{A}{\otimes} W).$$

Homotopy theory in $\mathbf{M}_A^W$

Under certain reasonable conditions on the monoidal model category $\mathbf{M}$ and on W, the forgetful functor

$$U_W : \mathbf{M}_A^W \to \mathbf{Mod}_A$$

left-induces a model category structure on $\mathbf{M}_A^W$.

Moreover,

$$\gamma_* : \mathbf{M}_A^W \to \mathbf{M}_A^{W'}$$

is then the left member of a Quillen pair, for all morphisms $\gamma : W \to W'$ of "nice enough" A-co-rings.

Example: The canonical co-ring I

Let $\varphi : B \to A$ be a morphism of monoids.

The canonical co-ring on φ, denoted W_φ, is $A \underset{B}{\otimes} A$, with comultiplication equal to the composite

$$A \underset{B}{\otimes} A \cong A \underset{B}{\otimes} B \underset{B}{\otimes} A \xrightarrow{A \underset{B}{\otimes} \varphi \underset{B}{\otimes} A} A \underset{B}{\otimes} A \underset{B}{\otimes} A \cong (A \underset{B}{\otimes} A) \underset{A}{\otimes} (A \underset{B}{\otimes} A).$$

The morphism $\bar{\mu} : A \underset{B}{\otimes} A \to A$ induced by the multiplication map of A is the counit.

Example: The canonical co-ring II

For any morphism of monoids $\varphi : B \to A$,

$$\mathbf{M}^{W_\varphi} \cong \mathbf{D}(\varphi),$$

the descent category associated to φ.

An object of $\mathbf{D}(\varphi)$ is a right A-module M endowed with a morphism $\theta : M \to M \underset{B}{\otimes} A$ such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\theta} & M \underset{B}{\otimes} A \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle \theta \underset{B}{\otimes} A} \\ M \underset{B}{\otimes} A & \xrightarrow{M \underset{B}{\otimes} \varphi \underset{B}{\otimes} A} & M \underset{B}{\otimes} A \underset{B}{\otimes} A \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\theta} & M \underset{B}{\otimes} A \\ & \searrow^{=} & \downarrow{\scriptstyle \bar{r}} \\ & & M \end{array}$$

commute.

The pair (M, θ) is a descent datum.

Canonical descent

Let $\varphi : B \to A$ be a morphism of monoids.

The canonical descent functor

$$\mathrm{Can} : \mathbf{Mod}_B \to \mathbf{D}(\varphi)$$

is defined on objects by $\mathrm{Can}(M) = (M \underset{B}{\otimes} A, \theta_M)$, with $\theta_M = M \underset{B}{\otimes} \varphi \underset{B}{\otimes} A$.

If $\mathbf{M}$ is a “nice enough” monoidal model category, then Can is the left member of a Quillen pair.

Example: Comodule algebras and co-rings

Let H be any bimonoid in $\mathbf{M}$.

Let A be an H-comodule algebra.

$A \otimes H$ is naturally an A-co-ring, with left A-action

$$A \otimes A \otimes H \xrightarrow{\mu_A \otimes H} A \otimes H,$$

and right A-action

$$A \otimes H \otimes A \xrightarrow{A \otimes H \otimes \rho} A \otimes H \otimes A \otimes H \xrightarrow{\cong} A \otimes A \otimes H \otimes H \xrightarrow{\mu_A \otimes \mu_H} A \otimes H$$

and comultiplication

$$A \otimes H \xrightarrow{A \otimes \Delta} A \otimes H \otimes H \cong (A \otimes H) \underset{A}{\otimes} (A \otimes H).$$

Henceforth, we denote this co-ring W_ρ.

Hopf-Galois data

- a bimonoid H
- a monoid B
- an H-comodule algebra A, with coaction $\rho : A \to A \otimes H$
- $\varphi : \mathrm{Triv}_H(B) \to A$ a morphism of H-comodule algebras

Assume that $\mathbf{M}$ and (H, B, A, φ) are "nice enough" to ensure the existence of all the necessary model category structures.

The Galois functor

The Galois map β_φ, which is equal to the composite

$$A \underset{B}{\otimes} A \xrightarrow{A \underset{B}{\otimes} \rho} A \underset{B}{\otimes} A \otimes H \xrightarrow{\mu \otimes H} A \otimes H,$$

underlies a morphism of A-co-rings, from W_φ to W_ρ.

The Galois map therefore induces

$$(\beta_\varphi)_* : \mathbf{D}(\varphi) \to \mathbf{M}_A^{W_\rho},$$

the Galois functor associated to φ, which is the left member of a Quillen pair, under reasonable conditions.

The corestriction functor

Let $j : A \xrightarrow{\sim} A'$ be a fibrant replacement in the category of H-comodule algebras.

A model for the homotopy coinvariants of the H-coaction on A is then

$$A^{hco\,H} := (A')^{co\,H}.$$

The homotopy corestriction map $i_\varphi : B \to A^{hco\,H}$ is equal to the composite

$$B \cong \big(\mathrm{Triv}_H(B)\big)^{co\,H} \xrightarrow{\varphi^{co\,H}} A^{co\,H} \xrightarrow{j^{co\,H}} (A')^{co\,H} = A^{hco\,H}$$

and induces a functor

$$i_\varphi^* : \mathbf{Mod}_{A^{hco\,H}} \to \mathbf{Mod}_B.$$

Reminder of the ring case

Recall that ...

...a homomorphism of rings $\varphi : B \to A$ is an H-Hopf-Galois extension if the Galois map

$$\beta_\varphi : A \underset{B}{\otimes} A \to A \underset{\Bbbk}{\otimes} H$$

and the corestriction map

$$i_\varphi : B \to A^{co\,H}$$

are both isomorphisms.

Definition

The morphism

$$\varphi : \mathrm{Triv}_H(B) \to A$$

of H-comodule algebras is a homotopic H-Hopf-Galois extension if the Galois functor

$$(\beta_\varphi)_* : \mathbf{D}(\varphi) \to \mathbf{M}_A^{W_\rho}$$

and the corestriction functor

$$i_\varphi^* : \mathbf{Mod}_{A^{hco\, H}} \to \mathbf{Mod}_B$$

are both Quillen equvalences.

Trivial extensions I

A bimonoid H is a Hopf monoid if the Galois functor

$$(\beta_\eta)_* : \mathbf{D}(\eta) \to M_H^{W_\Delta}$$

associated to the H-comodule algebra map $\eta : \mathrm{Triv}(I) \to H$ is a Quillen equivalence.

Examples

The monoid of Moore loops on a topological space is a Hopf monoid in **Top**.

Any bialgebra in the category of chain complexes over a commutative ring is a Hopf monoid.

Trivial extensions II

Proposition

If H is a Hopf monoid and B is a fibrant monoid such that $B \otimes -$ preserves weak equivalences, then

$$B \xrightarrow{B \otimes \eta} B \otimes H$$

is a homotopic H-Hopf-Galois extension.

Example: Simplicial monoids

Let H be a simplicial monoid, seen as a simplicial bimonoid, via the diagonal map.

Let A be a fibrant H-comodule algebra, i.e., a simplicial monoid endowed with a simplicial homomorphism $p : A \to H$ that is a Kan fibration.

Let B be a simplicial monoid, and let $\varphi : \mathrm{Triv}_H(B) \to A$ be a morphism of H-comodule algebras.

Proposition

φ is a homotopic H-Hopf-Galois extension iff it is homotopy equivalent to a principal fibration of simplicial monoids.

Example: Chain algebras I

Let H be a 1-connected bialgebra in the category of finite-type chain complexes of $\mathbb{k}$-vector spaces.

Let A be a connected H-comodule algebra, with H-coaction ρ.

Proposition

The algebra map

$$A \to \Omega(A; H; H) : a \mapsto a_i \otimes 1 \otimes h^i,$$

where $\rho(a) = a_i \otimes h^i$, is a fibrant replacement of A as an H-comodule algebra.

- [H.-Levi] $\Omega(A; H; H)$ admits an algebra structure.
- Fibrancy of $\Omega(A; H; H)$ proved by showing that it is the limit of a “Postnikov tower.”

Example: Chain algebras II

Example

The algebra map induced by the unit of H

$$\iota : \Omega(A; H; \Bbbk) \hookrightarrow \Omega(A; H; H)$$

is a homotopic H-Hopf-Galois extension.

Remark

$$\Omega(A; H; H) \underset{\Omega(A;H;\Bbbk)}{\otimes} \Omega(A; H; H) \cong \Omega(A; H; H) \otimes H$$

and

$$\Omega(A; H; H)^{hco\, H} = \Omega(A; H; H)^{co\, H} \cong \Omega(A; H; \Bbbk),$$

since $\Omega(A; H; H)$ is fibrant. Thus, both i_φ and β_φ are actually isomorphisms in this case.

Goal

To prove a structure theorem relating the notions of

- homotopic Hopf-Galois extensions,
- homotopical faithful flatness, and
- descent,

analogous to a well-known and important theorem in the ring case, due to Schneider.

Induction

Let H be a bimonoid, and let A be an H-comodule algebra with H-coaction map ρ.

The ρ-induction functor

$$\mathrm{Ind}_\rho : \mathbf{Mod}_{A^{co\,H}} \to \mathbf{M}_A^{W_\rho}$$

is defined on objects by

$$\mathrm{Ind}_\rho(M) = (M \underset{A^{co\,H}}{\otimes} A, M \underset{A^{co\,H}}{\otimes} \rho).$$

If $\mathbf{M}$ is a "nice enough" monoidal model category, then Ind_ρ is the left member of a Quillen pair.

Schneider's structure theorem

Theorem

Let $\Bbbk$ be a commutative ring, and let H be a $\Bbbk$-flat Hopf algebra.

The following are equivalent for any H-comodule algebra A, with coinvariant algebra $B = A^{co\,H}$.

- *The inclusion $B \hookrightarrow A$ is an H-Hopf-Galois extension, and A is a faithfully flat B-module.*
- *The functor $\mathrm{Ind}_\rho : \mathbf{Mod}_B \to \mathbf{M}_A^{W_\rho}$ is an equivalence, where ρ denotes the H-coaction on A.*

(A is faithfully flat over B if A is flat over B and $M \underset{B}{\otimes} A = 0 \Rightarrow M = 0$.)

Characterizing faithful flatness

Theorem

Let $\varphi : B \to A$ be an inclusion of rings.

The canonical descent functor

$$\mathrm{Can} : \mathbf{Mod}_B \to \mathbf{D}(\varphi)$$

is an equivalence of categories if and only if A is faithfully flat as a B-module.

Homotopical faithful flatness

Let $\varphi : B \to A$ be a morphism of monoids in **M**.

The monoid A is homotopically faithfully flat over B if

$$\mathrm{Can} : \mathbf{Mod}_B \to \mathbf{D}(\varphi)$$

is the left member of a Quillen equivalence.

The homotopical structure theorem

Theorem

Let $\mathbf{M}$ *be a monoidal model category. Let* $(H, B, \mathrm{Triv}_H(B) \xrightarrow{\varphi} A)$ *be Hopf-Galois data.*

Under reasonable conditions on $\mathbf{M}$ *and on the Hopf-Galois data, the following conditions are equivalent.*

- *The monoid map* φ *is a homotopic* H*-Hopf-Galois extension, and* A *is homotopically faithfully flat over* B*.*
- *The functor*

$$\mathrm{Ind}_\rho \circ (- \underset{B}{\otimes} A^{co\,H}) : \mathbf{Mod}_B \to \mathbf{M}_A^{W_\rho}$$

is a Quillen equivalence.