A metric Kan-Thurston theorem

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I’ll describe a new(ish) proof of the Kan-Thurston theorem and some other related results, using CAT(0) cubical complexes.

1. An introduction to CAT(0) metric spaces;
2. Around the Kan-Thurston theorem;
3. My version and corollaries;
4. Remarks on the proof.

*These slides differ slightly from the talk. They now cite a result of Wegner that I did not know until after the talk to strengthen one of the corollaries. Also included are 10 slides that I did not have time for during the talk.*
CAT(0) geometry

A geodesic metric space is a metric space in which any two points are joined by a path of length equal to the distance between them.

A CAT(0)-metric space is a geodesic metric space in which each geodesic triangle is at least as thin as a triangle with the same side lengths in the Euclidean plane.
Properties of CAT(0) spaces

There is a unique geodesic between any two points.

CAT(0) spaces are contractible: for any point $x_0$, pulling points towards $x_0$ along geodesics defines a contraction.

If a group $G$ acts by isometries on a CAT(0) space $X$ then the fixed point set $X^G$ is either contractible or empty.

A Cartan-Hadamard theorem: if $X$ is locally CAT(0), then the universal cover of $X$ is CAT(0) (and hence contractible).
The length of a PL path in a cubical complex is measured by using the standard (side length 1) Euclidean metric on each cube.

The path metric $d(x, y)$ between two points of a cubical complex is the infimum of the lengths of the PL paths from $x$ to $y$.

This infimum is attained.
Gromov’s criterion

A cubical complex, with the path metric, is locally CAT(0) if and only if all vertex links are flag complexes.

The solid cube and the plane are locally CAT(0), the hollow cube is not.
The Kan-Thurston theorem

Theorem (Kan-Thurston)

For any path-connected topological space $X$, there is a discrete group $G_X$ and a map $t_X : K(G_X, 1) \to X$ which is a homology isomorphism.

The first step is to replace $X$ by $\text{Sing}_\bullet(X)$, the complex whose $n$-simplices are all continuous maps $\Delta^n \to X$. Thus the theorem is really a result about simplicial complexes.

Theorem (Baumslag-Dyer-Heller)

For any simplicial complex $X$, there is $G_X$ as before. Also, if $X$ is finite, then $K(G_X, 1)$ may be taken to be finite too.
Variations

**Theorem (Hausmann)**

*In the case when $X$ is finite, $G_X$ can be taken to be a duality group.*

**Theorem (McDuff)**

*For any simplicial complex $X$, there is a discrete monoid $M$ and a map $t_M : BM \to X$ which is a homotopy equivalence.*

**Theorem (L-Nucinkis)**

*For any simplicial complex $X$, there is a discrete group $H = H_X$ and a map $t_X : E_H/H \to X$ which is a homotopy equivalence. Moreover, $H$ contains a torsion-free subgroup of index 2.*
Methods

The proofs of McDuff’s theorem and the theorem of L-Nucinkis follow the pattern described by Baumslag-Dyer-Heller and by Maunder.

Proving the Kan-Thurston theorem relies on having a source of *acyclic groups*: groups $G$ for which a $K(G, 1)$ has the same homology as a point.

The other proofs rely on having sources of monoids $M$ for which $BM \sim \ast$ and groups $G$ for which $EG/G \sim \ast$.

The other results *look* stronger, because you can find monoids $M$, and groups $G$ for which $BM$ and $EG/G$ are contractible. In contrast, $BG = K(G, 1) \sim \ast$ happens only when $G$ is the trivial group.
The new result

Theorem

For every simplicial complex $X$ there is a locally CAT(0) cubical complex $T_X$, a map $t_X : T_X \to X$ and a (cellular, isometric) involution $\tau$ on $T_X$ such that

- $t_X : T_X \to X$ is a homology isomorphism;
- $t_X \circ \tau = t_X$ and the map $T_X/\tau \to X$ is a homotopy equivalence;
- $t_X : T^\tau_X \to X$ is a homology isomorphism;
- If $X$ is finite, so is $T_X$;
- The construction is natural. If $Y$ is a subcomplex of $X$, then $T_Y$ is a totally geodesic subcomplex of $T_X$. 
Remarks

Ignoring the involution $\tau$, the group $G_X = \pi_1(T_X)$ is a Kan-Thurston group for $X$.

Let $H_X$ be the index-two supergroup of $G_X$ defined as the self-maps of $\tilde{T}_X$ that lift either $I$ or $\tau$.

$\tilde{T}_X$ is a model for $EH_X$, and so

$$EH_X/H_X = \tilde{T}_X/H_X = T_X/\tau \sim X.$$  

For simplicity, let’s ignore $\tau$ for now.
Theorem
For every simplicial complex $X$ there is a locally $\text{CAT}(0)$ cubical complex $T_X$, and a map $t_X : T_X \to X$ such that

- $t_X : T_X \to X$ is a homology isomorphism;
- If $X$ is finite, so is $T_X$;
- The construction is natural for simplicial maps of $X$ that are injective on each simplex. If $Y$ is a subcomplex of $X$, then $T_Y$ is a totally geodesic subcomplex of $T_X$. 
Corollaries I

Corollary

Suppose that \( G \) acts with finite stabilizers on connected \( X \). There is a group \( \tilde{G} \), a surjection \( \tilde{G} \to G \) and an equivariant homology isomorphism \( E\tilde{G} \to X \).

Proof

\( G \) also acts on \( T_X \). Let \( \tilde{T} \) be the universal cover of \( T_X \), and let \( \tilde{G} \) be the group of all self-maps of \( \tilde{T} \) that lift the action of some element of \( G \) on \( T_X \).

\( \tilde{G} \) acts on the CAT(0) cubical complex \( \tilde{T} \) with finite stabilizers, and so \( \tilde{T} \) is a model for \( E\tilde{G} \).
Corollary

*For any group* \( G \), there is a group \( \tilde{G} \rightarrow G \) such that

- \( \tilde{G} \) acts with finite stabilizers on a CAT(0) cubical complex;

- for every equivariant homology theory \( \mathcal{K}^\ast \), the map \( \mathcal{K}^\tilde{G}_\ast(EG) \rightarrow \mathcal{K}^G_\ast(EG) \) is an isomorphism.

**Proof**

Take \( X = EG \) in the previous corollary.
Corollary

For every group $G$, there is a group $\tilde{G}$ and a surjection $\tilde{G} \to G$ such that for any field $\mathbb{F}$ of characteristic zero, the Farrell-Jones assembly conjecture for the algebraic $K$-theory of $\mathbb{F}G$ holds if and only if the natural map $K_*(\mathbb{F}\tilde{G}) \to K_*(\mathbb{F}G)$ is an isomorphism.

Proof
Take $\tilde{G}$ as in the previous corollary. Wegner has shown that the Farrell-Jones conjecture holds for $\mathbb{F}G$, extending work of Bartels-Lück.
A related result

**Theorem (Raeyong Kim)**

For any finite $X$, there is a locally CAT(0) cubical complex $V_X$ such that $\pi_1(V_X)$ is a duality group and there is a homology isomorphism $\nu : V_X \to X$.

**Proof**

Along the same lines as Hausmann’s proof. Uses work of Brady-Meier to establish that $\pi_1(V_X)$ is a duality group.
Suppose we just want, for each $n \geq 1$, a group $G_n$ with the same homology as $S^n$.

B-D-H take a fixed acyclic group $H$ with a finite $BH$; this group will play the role of the 2-disc.

Can take $G_1 = \mathbb{Z}$. Now we make a 2-sphere by gluing two 2-discs along a circle: $G_2 = H \ast_{G_1} H$.

For the 3-disc, consider the group $H_3 = H \ast_{G_1} (G_1 \times H)$. This contains a copy of $G_2$.

Inductively, $H_{n+1} = H_n \ast_{G_{n-1}} (G_{n-1} \times H)$ and $G_{n+1} = H_{n+1} \ast_{G_n} H_{n+1}$.

This all works fine in the CAT(0) world, except we need to find a suitable $H$. 
An acyclic locally CAT(0) 2-complex

Consider the group with generators $a, b, c, d, e, f$ and relators

$$abcb^{-1}eb, \quad adcd^{-1}ed, \quad afcf^{-1}ef,$$
$$a^2b^{-1}cbeb^{-1}, \quad ad^{-1}c^2ded^{-1}, \quad af^{-1}cf^2ef^{-1}.$$

The matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
2 & 1 & 1 & -1 & 0 & 0 \\
1 & 2 & 1 & 0 & -1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

is easily seen to be invertible over $\mathbb{Z}$, and so the presentation 2-complex is acyclic.
An acyclic locally CAT(0) 2-complex II

We can build the presentation 2-complex by gluing six hexagons onto a 6-petalled rose. Take 3 hexagons of each of these two types:

These relations need the left-hand hexagon:

\[ abcb^{-1}eb, \quad adcd^{-1}ed, \quad afcf^{-1}ef, \]

and these need the right-hand one:

\[ a^2b^{-1}cbeb^{-1}, \quad ad^{-1}c^2ded^{-1}, \quad af^{-1}cf^2ef^{-1}. \]
An acyclic locally CAT(0) 2-complex III

To verify that this square complex is locally CAT(0), we look at the links. The only complicated one is that of the vertex at the centre of the rose:
The involution $\tau$

The slides from this point on were not shown during the talk.

Any acyclic locally CAT(0) 2-complex (such as the one on the three previous slides) can be used to prove a CAT(0) version of the Kan-Thurston theorem. To get the full version of the main theorem, we use a more complicated construction.

We use a pair of acyclic locally CAT(0)-complexes $(A', A)$ with the following properties:

- $A$ is a totally geodesic subcomplex of $A'$;
- $A'$ has an involution $\tau$ with $A$ as its fixed-point set;
- $A'/\tau$ is contractible.
Another acyclic locally CAT(0) 2-complex

For $i \in \mathbb{Z}/4$, let $A_i = a_i a_{i+2} a_i^{-2} a_{i+2}^{-1} a_i$.

Similarly, let $B_i = b_i b_{i+2} b_i^{-2} b_{i+2}^{-1} b_i$.

Consider the eight words

$a_i A_i B_i A_{i+1} B_i A_{i+2} B_i A_{i+3} B_i$

$b_i B_i A_i^{-1} B_{i+1} A_i^{-1} B_{i+2} A_i^{-1} B_{i+3} A_i^{-1}$

The presentation 2-complex $A$ is acyclic, and $\mathbb{Z}/4$ acts on $A$.

Let $t$ be the element of order 2 in $\mathbb{Z}/4$.

$A^t$ is a single point, and $A/t$ is contractible.
Another acyclic locally CAT(0) 2-complex II

We can build a CAT(0) octagon out of unit squares such that

- Each side is a geodesic
- Each corner is a right-angle
- The side lengths are in the ratios $(7, 6, 6, 6, 6, 6, 6, 6)$. 

![Octagon diagram]
Another acyclic locally CAT(0) 2-complex III

Use these CAT(0) octagons to build the presentation 2-complex $P$:

$$a_i A_i B_i A_{i+1} B_i A_{i+2} B_i A_{i+3} B_i$$

$$b_i B_i A_i^{-1} B_{i+1} A_i^{-1} B_{i+2} A_i^{-1} B_{i+3} A_i^{-1}$$

Every possible meeting between an end of an $a_i$ and an end of a $b_j$ occurs exactly once.

Other meetings between letters are between ends of letters of the same colour. Since the word is reduced, an end of a letter never meets itself.
Another acyclic locally CAT(0) 2-complex IV

The presentation 2-complex is therefore locally CAT(0), since the link of the main vertex is as below.
We want to embed $A \subset A'$, where $A'$ is acyclic and admits an involution $\tau$ such that $A = A'^{\tau}$ and $A'/\tau$ is contractible.

For this, we use the involution $t$ on $A$.

We could use $A' = A \times A$, with $\tau$ defined by swapping the factors, but then $A$ would not be a cubical subcomplex.

The picture below represents $A$, with the fixed point for $t$ marked in red, and the 1-skeleton loops $a_1$ and $b_1$ in bold.
An acyclic locally CAT(0) 3-complex II

The picture below represents $A \times [-4, 4]$, with the fixed point set for the involution $(a, x) \mapsto (t(a), -x)$ in red.
An acyclic locally CAT(0) 3-complex III

The picture below represents the mapping torus of $t : A \rightarrow A$, made by identifying the ends of $A \times [-4, 4]$. The involution on the previous slide defines an involution on this mapping torus.
Now identify the points \((a_0, 0)\) and \((a_0, 4)\). This gives a space with an involution with fixed point set \(A\) and the same homology as a 2-petalled rose.
An acyclic locally CAT(0) 3-complex V

We can kill the homology of this without changing the fixed point set by gluing another copy of $A$ along the rose marked in green.

The resulting space is $A'$, with $A \subseteq A'$ shaded in red.