Algebraic $K$-theory
Exercise Set 5

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1. Let $R$ be a ring, and let $\mathcal{I}(R)$ denote the set of all two-sided ideals of $R$. Show that $(\mathcal{I}(R), +, 0, *)$ is a semiring, where $I + J = \{x + y \mid x \in I, y \in J\}$ and $I * J = \{xy \mid x \in I, y \in J\}$ for all $I, J \in \mathcal{I}(R)$, and calculate its group completion.

2. Let $X$ be any set, and let $\mathcal{P}(X)$ denote the power set of $X$. Show that both $(\mathcal{P}(X), \cap, X, \cup)$ and $(\mathcal{P}(X), \cup, \emptyset, \cap)$ are commutative semirings, and calculate their group completions.

3. Prove the following important, elementary properties of the tensor product construction. Let $R, S, T$ and $U$ be any rings.

   (a) If $M \in R\text{Mod}_S$ and $N \in S\text{Mod}_T$, then $M \otimes_S N \in R\text{Mod}_T$.

   (b) If $M \in R\text{Mod}_S$, then $R \otimes_R M \cong M$ as $(R, S)$-bimodules.

   (c) Let $\mathcal{J}$ be any set. If $M_j \in R\text{Mod}_S$ for all $j \in \mathcal{J}$, and $N \in S\text{Mod}_T$, then
      $$( \bigoplus_{j \in \mathcal{J}} M_j ) \otimes_S N \cong \bigoplus_{j \in \mathcal{J}} (M_j \otimes_S N)$$
      as $(R, T)$-bimodules

   (d) If $L \in R\text{Mod}_S$, $M \in S\text{Mod}_T$ and $N \in T\text{Mod}_U$, then
      $$(L \otimes_S M) \otimes_T N \cong L \otimes_S (M \otimes_T N)$$
      as $(R, U)$-bimodules.