Algebraic K-theory
Exercise Set 5

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1. Let R be a ring, and let $\mathcal{I}(R)$ denote the set of all two-sided ideals of R. Show that $(\mathcal{I}(R), +, 0, *)$ is a semiring, where $I+J = \{x+y \mid x \in I, y \in J\}$ and $I * J = \{xy \mid x \in I, y \in J\}$ for all $I, J \in \mathcal{I}(R)$, and calculate its group completion.

2. Let X be any set, and let $\mathfrak{P}(X)$ denote the power set of X. Show that both $(\mathfrak{P}(X), \cap, X, \cup)$ and $(\mathfrak{P}(X), \cup, \emptyset, \cap)$ are commutative semirings, and calculate their group completions.

3. Prove the following important, elementary properties of the tensor product construction. Let R, S, T and U be any rings.

   (a) If $M \in {}_R\mathbf{Mod}_S$ and $N \in {}_S\mathbf{Mod}_T$, then $M \otimes_S N \in {}_R\mathbf{Mod}_T$.

   (b) If $M \in {}_R\mathbf{Mod}_S$, then $R \otimes_R M \cong M$ as (R, S)-bimodules.

   (c) Let $\mathcal{J}$ be any set. If $M_j \in {}_R\mathbf{Mod}_S$ for all $j \in \mathcal{J}$, and $N \in {}_S\mathbf{Mod}_T$, then
   $$\Big(\bigoplus_{j \in \mathcal{J}} M_j\Big) \otimes_S N \cong \bigoplus_{j \in J} (M_j \otimes_S N)$$
   as (R, T)-bimodules

   (d) If $L \in {}_R\mathbf{Mod}_S$, $M \in {}_S\mathbf{Mod}_T$ and $N \in {}_T\mathbf{Mod}_U$, then
   $$(L \otimes_S M) \otimes_T N \cong L \otimes_S (M \otimes_T N)$$
   as (R, U)-bimodules.