Algebraic K-theory
Exercise Set 6

08.04.2011

1. Let R be a commutative ring such that $K_0(R)$ is a cyclic group. Prove that the only idempotents of R are 0 and 1. What if R is noncommutative?

2. Let C_2 denote the cyclic group of order 2. Compute $K_0\big(\mathbb{Q}[C_2]\big)$.

3. Apply restriction of scalars to proving that if R and S are isomorphic rings, then $K_0(R)$ and $K_0(S)$ are isomorphic abelian groups.

4. (A first glimpse of the Morita invariance of K_0.) Let R be any ring. Prove that
$$K_0\big(\operatorname{Mat}_{m,m}(R)\big) \cong K_0\big(\operatorname{Mat}_{n,n}(R)\big)$$
as abelian groups for all positive integers m and n. In particular,
$$K_0(R) \cong K_0\big(\operatorname{Mat}_{n,n}(R)\big)$$
for all n.

   Hint: Observe that $\operatorname{Mat}_{k,l}(R) \in {}_{\operatorname{Mat}_{k,k}(R)}\mathbf{Mod}_{\operatorname{Mat}_{l,l}(R)}$ for all k, l, and prove that
$$\operatorname{Mat}_{k,l}(R) \otimes_{\operatorname{Mat}_{l,l}(R)} \operatorname{Mat}_{l,m}(R) \cong \operatorname{Mat}_{k,m}(R)$$
for all k, l, m.

5. Let R be a ring with IBN, and let $R' = \operatorname{Mat}_{n,n}(R)$. Prove that if $n > 1$, then the short exact sequence
$$0 \to K_0(\mathbb{Z}) \xrightarrow{K_0(\eta)} K_0(R') \to \widetilde{K}_0(R') \to 0$$
does not split.

   Hint: Prove that $[R']$ is divisible by n in $K_0(R')$.