1. Let $R$ be a commutative ring such that $K_0(R)$ is a cyclic group. Prove that the only idempotents of $R$ are 0 and 1. What if $R$ is noncommutative?

2. Let $C_2$ denote the cyclic group of order 2. Compute $K_0(\mathbb{Q}[C_2])$.

3. Apply restriction of scalars to proving that if $R$ and $S$ are isomorphic rings, then $K_0(R)$ and $K_0(S)$ are isomorphic abelian groups.

4. (A first glimpse of the Morita invariance of $K_0$.) Let $R$ be any ring. Prove that

$$K_0(\text{Mat}_{m,m}(R)) \cong K_0(\text{Mat}_{n,n}(R))$$

as abelian groups for all positive integers $m$ and $n$. In particular,

$$K_0(R) \cong K_0(\text{Mat}_{n,n}(R))$$

for all $n$.

**Hint:** Observe that $\text{Mat}_{k,l}(R) \in \text{Mat}_{k,k}(R) \text{Mod}_{\text{Mat}_{l,l}(R)}$ for all $k, l$, and prove that

$$\text{Mat}_{k,l}(R) \otimes_{\text{Mat}_{l,l}(R)} \text{Mat}_{l,m}(R) \cong \text{Mat}_{k,m}(R)$$

for all $k, l, m$.

5. Let $R$ be a ring with IBN, and let $R' = \text{Mat}_{n,n}(R)$. Prove that if $n > 1$, then the short exact sequence

$$0 \to K_0(\mathbb{Z}) \xrightarrow{K_0(n)} K_0(R') \to \tilde{K}_0(R') \to 0$$

does not split.

**Hint:** Prove that $[R']$ is divisible by $n$ in $K_0(R')$. 
