1. Prove that $K_1(\mathcal{F}(-))$ and $K_1(\mathcal{P}(-))$ are natural, i.e., that they extend to functors $K_1(\mathcal{F}(-)), K_1(\mathcal{P}(-)) : \text{Ring} \to \text{Ab}$. Show that the isomorphisms $K_1 R \cong K_1(\mathcal{F}(R)) \cong K_1(\mathcal{P}(R))$ seen in class are natural.

2. Let $C$ be a modest subcategory of $R\text{Mod}$, and let $M \in C$, $\alpha \in \text{Aut}(M)$. If $M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n$ is a $C$-filtration of $M$ such that $\alpha(M_i) = M_i$ for all $i$, prove that

$$[M, \alpha] = \sum_{i=0}^{n-1} [M_i/M_{i+1}, \alpha_i]$$

in $K_1(C)$, where $\alpha_i \in \text{Aut}(M_i/M_{i+1})$ is induced by $\alpha$.

3. Let $C$ be a modest subcategory of $R\text{Mod}$ such that if $f : M \to N$ is a morphism in $C$, then $\ker f$ is an object in $C$. Let

$$0 \to (M_n, \alpha_n) \to \cdots \to (M_0, \alpha_0) \to 0$$

be an exact sequence in $\text{Aut}(C)$. Prove that

$$\sum_{i=0}^{n} (-1)^i [M_i, \alpha_i] = 0.$$

4. (A very brief introduction to the Whitehead group, which plays a crucial role in geometric topology.) Let $G$ be a group, and let $w$ denote the composite homomorphism

$$G \times \{1, -1\} \xrightarrow{j} GL_1(\mathbb{Z}[G]) \xrightarrow{s_1} K_1(\mathbb{Z}[G]),$$

where $j(g, \pm 1) = (\pm g)$. The Whitehead group of $G$ is

$$Wh(G) := \text{coker} w = K_1(\mathbb{Z}[G]) / \langle s_1(\pm g) \mid g \in G \rangle.$$
(a) Prove that $Wh\{e\} \cong \{e\}$.

(b) Let $C_2$ denote the cyclic group of order 2. Prove that

$$Wh(C_2) \cong SK_1(D(\mathbb{Z}, (2))).$$

(c) (A bit tricky; see [Rosenberg, Thm. 2.4.3].) Prove that $SK_1(D(\mathbb{Z}, (2)))$ is in fact the trivial group, whence $Wh(C_2)$ is trivial.

(d) Let $C_5$ denote the cyclic group of order 5. Show that there is an element of infinite order in $Wh(C_5)$.

**Hint:** Let $t$ denote the generator of $C_5$. Show that $a = 1 - t - t^{-1}$ is a unit in $\mathbb{Z}[G]$. Next show that the composite of the homomorphism $\alpha : \mathbb{Z}[G] \to \mathbb{C}$ specified by $\alpha(t) = e^{i2\pi/5}$ with the norm map $\mathbb{C} \to \mathbb{R}_{\geq 0}$ induces a homomorphism $Wh(C_5) \to (\mathbb{R}_{>0}, \cdot)$.

(e) Let $G_{ab}$ denote the abelianization of $G$. Prove that there are group homomorphisms

$$G_{ab} \times \{1, -1\} \xrightarrow{j} K_1(\mathbb{Z}[G]) \xrightarrow{r} G_{ab} \times \{1, -1\}$$

such that $rj = \text{Id}$. Conclude that

$$K_1(\mathbb{Z}[G]) \cong G_{ab} \times \{1, -1\} \times Wh(G).$$

**Remark 0.1.** In fact,

$$Wh(C_n) = \begin{cases} \{e\} & : n = 2, 3, 4, 6 \\ \mathbb{Z} & : n = 5. \end{cases}$$

More generally, if $G$ is finite abelian, then the determinant induces an isomorphism

$$\det : K_1(\mathbb{Z}[G]) \to \mathbb{Z}[G]^*$$

and so

$$Wh(G) \cong \mathbb{Z}[G]^*/\langle \pm g \mid g \in G \rangle.$$ 

On the other hand, Bass, Heller and Swan proved that if $G$ is free abelian, then $Wh(G)$ is always trivial.