Elements of Homotopy **Fall 2008**
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Series 11

Let Y be a space and recall that a continuous map $p : E \longrightarrow B$ satisfies the *homotopy lifting property* with respect to Y if given any solid commutative diagram of the form

(1)
$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{g} & E \\ \subseteq \downarrow & \overset{\xi}{\nearrow} \exists & \downarrow p \\ Y \times I & \xrightarrow[H]{} & B \end{array}$$

in Top, there exists a continuous map ξ such that (1) commutes.

Let F be space and recall that a *locally trivial bundle* with fiber F is a continuous map $p : E \longrightarrow B$ such that for each $b \in B$ there is a neighborhood $U_\alpha \subseteq B$ of b and a homeomorphism $\varphi_\alpha : U_\alpha \times F \longrightarrow p^{-1}(U_\alpha)$ which makes the diagram

$$\begin{array}{ccc} U_\alpha \times F & \xrightarrow[\cong]{\varphi_\alpha} & p^{-1}(U_\alpha) \\ \downarrow \mathrm{pr}_1 & & \downarrow p| \\ U_\alpha & = & U_\alpha \end{array}$$

commute; the open sets U_α are sometimes called *coordinate neighborhoods*. Here, pr_1 denotes the projection map (onto the first factor) and $p|$ denotes the indicated restriction of p. Note that every vector bundle is a locally trivial bundle.

Exercise 1.

(a) Let F, B be spaces. Prove that the projection map $\mathrm{pr}_1 : B \times F \longrightarrow B$ is a locally trivial bundle with fiber F. This is sometimes called a *trivial bundle* (or product bundle).

(b) Prove that a covering map $p : E \longrightarrow B$ with path connected base B is a locally trivial bundle with discrete fiber. Conclude that

$$\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1,$$
$$\mathbb{Z}^2 \longrightarrow \mathbb{R}^2 \longrightarrow T,$$
$$\mathbb{Z}/2\mathbb{Z} \longrightarrow S^n \longrightarrow \mathbb{R}P^n,$$

are locally trivial bundles of the form $F \longrightarrow E \longrightarrow B$.

Exercise 2. Let $n \geq 1$. Define the space $\mathbb{C}P^n := (\mathbb{C}^{n+1} - \{0\})/\sim$ such that $\sim$ is generated by the relations $z \sim \lambda z$, for each $\lambda \in \mathbb{C} - \{0\}$ and $z \in \mathbb{C}^{n+1} - \{0\}$. Hence $\mathbb{C}P^n$, which is called *n-dimensional complex projective space*, is the set of all lines through the origin in $\mathbb{C}^{n+1}$, equipped with the quotient topology. Define a map p by restricting the natural projection map $\mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}P^n$ to the subspace S^{2n+1} as indicated

$$\begin{array}{ccc} \mathbb{C}^{n+1} - \{0\} & \longrightarrow & \mathbb{C}P^n \\ \subseteq \uparrow & \overset{p}{\nearrow} & \\ S^{2n+1} & & \end{array}$$

in the diagram. The purpose of this exercise is to prove that $p : S^{2n+1} \longrightarrow \mathbb{C}P^n$ is a locally trivial bundle with fiber S^1.

(a) Define $U_i := \{[z] \in \mathbb{C}P^n \mid z_i \neq 0\}$ and verify that the collection of sets $U_0, \cdots, U_n$ is an open cover of $\mathbb{C}P^n$. Here, $[z]$ is notation for the equivalence class of $z = (z_0, \cdots, z_n) \in \mathbb{C}^{n+1} - \{0\}$.
(b) Prove that $p^{-1}(U_i) = \{z \in \mathbb{C}^{n+1} - \{0\} |\ |z|^2 = 1,\ z_i \neq 0\}$.
(c) Define

$$\varphi_i : U_i \times S^1 \longrightarrow p^{-1}(U_i), \qquad ([z], \lambda) \longmapsto \frac{z}{|z|} \frac{\lambda |z_i|}{z_i},$$
$$\psi_i : p^{-1}(U_i) \longrightarrow U_i \times S^1, \qquad z \longmapsto \big([z], \frac{z_i}{|z_i|}\big),$$

and prove that these maps are well-defined, continuous, and satisfy $\varphi_i \psi_i = \mathrm{id}$ and $\psi_i \varphi_i = \mathrm{id}$.
(d) Conclude that

(2) $$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n \qquad (n \geq 1)$$
(3) $$S^1 \longrightarrow S^3 \longrightarrow S^2 \qquad (n = 1)$$

are locally trivial bundles of the form $F \longrightarrow E \longrightarrow B$. The locally trivial bundle in (2) is called the *Hopf bundle* (or Hopf fibration). To obtain the special case (3), we used the fact that $\mathbb{C}P^1 \cong S^2$; see, for example, [1, Section 11.5.6].

Remark 3. It is proved in [4, Section 2.7] that if $p : E \longrightarrow B$ is a locally trivial bundle with a paracompact Hausdorff base B, then p has the homotopy lifting property with respect to every space Y, and hence such a p is a fibration. In particular, every vector bundle with a paracompact Hausdorff base is a fibration.

We will content ourselves with the following, which we will soon see is enough for constructing an associated long exact sequence of homotopy groups.

Theorem 4. *Let $p : E \longrightarrow B$ be a locally trivial bundle with fiber F. If Y is a compact Hausdorff space, then p has the homotopy lifting property with respect to Y.*

We will prove this theorem in Exercise 6. First we recall the following proposition which is proved in [2, Chapter 11].

Proposition 5. *Let Y be a compact Hausdorff space. Then given any open cover $\{V_\alpha\}$ of $Y \times I$ there exist closed subspaces Y_k of the form*

$$Y \times \{0\} =: Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ \cdots \ \subseteq Y_n := Y \times I$$

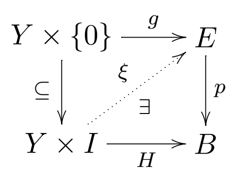
such that

(a) $\overline{Y_k \backslash Y_{k-1}} \subseteq V_\alpha$ *for some* α, *and*
(b) $Y_{k-1} \cap \overline{Y_k \backslash Y_{k-1}}$ *is a retract of* $\overline{Y_k \backslash Y_{k-1}}$.

Here, $\overline{Y_k \backslash Y_{k-1}}$ *denotes the closure of* $Y_k \backslash Y_{k-1}$ *in* Y_n.

Exercise 6. Let Y be a compact Hausdorff space and $p : E \longrightarrow B$ be a locally trivial bundle with fiber F. Consider any solid commutative diagram of the form

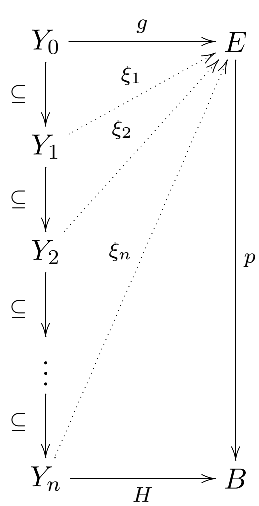
(4)



in Top. The purpose of this exercise is to prove that a lift ξ exists.

(a) Use Proposition 5 to construct a sequence of maps $\xi_1, \xi_2, \cdots, \xi_n$ which solve the sequence of lifting problems indicated in diagram (5) below.
(b) Conclude that taking $\xi = \xi_n$ makes the diagram (4) commute.

(5)



To get started with the construction of ξ_1, note that by Proposition 5 there exists a coordinate neighborhood U_α such that $H(\overline{Y_1 \backslash Y_0}) \subseteq U_\alpha$. Note also that the diagram

$$\begin{array}{ccc} Y_0 \cap \overline{Y_1 \backslash Y_0} & \xrightarrow{\subseteq} & \overline{Y_1 \backslash Y_0} \\ {\scriptstyle \subseteq}\big\downarrow & & \big\downarrow{\scriptstyle \subseteq} \\ Y_0 & \xrightarrow{\subseteq} & Y_0 \cup \overline{Y_1 \backslash Y_0} = Y_1 \end{array}$$

is a pushout square, since all the indicated subspaces are closed in Y_1. Hence, giving a map $Y_1 \longrightarrow E$ is the same as giving two maps

$$Y_0 \longrightarrow E, \qquad \overline{Y_1 \backslash Y_0} \longrightarrow E$$

which agree on $Y_0 \cap \overline{Y_1 \backslash Y_0}$.

The following is a generalization of the unique path lifting property for covering maps. We will prove this theorem in Exercises 8 and 9.

Theorem 7. *Let $p : E \longrightarrow B$ be a covering map. Given any solid commutative diagram of the form*

(6)
$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{g} & E \\ {\scriptstyle \subseteq}\big\downarrow & \overset{\xi}{\nearrow} \; {\scriptstyle \exists!} & \big\downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow[H]{} & B \end{array}$$

in Top*, there exists a unique continuous map ξ such that* (6) *commutes.*

In particular, a covering map satisfies the homotopy lifting property with respect to every space Y. Hence, a covering map is a fibration.

Exercise 8. Let $p : E \longrightarrow B$ be a covering map and consider any solid commutative diagram of the form

(7)
$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{g} & E \\ {\scriptstyle \subseteq}\big\downarrow & \overset{\xi}{\nearrow}\!\underset{\xi'}{\nearrow} & \big\downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow[H]{} & B \end{array}$$

in Top. Suppose there exist two maps ξ, ξ' such that diagram (7) commutes.

(a) Use unique path lifting to prove that $\xi = \xi'$.

This can be argued by considering the following diagram

$$\begin{array}{ccccccc} \{0\} & \xrightarrow{\cong} & \{y\} \times \{0\} & \xrightarrow{\subseteq} & Y \times \{0\} & \xrightarrow{g} & E \\ {\scriptstyle \subseteq}\big\downarrow & & {\scriptstyle \subseteq}\big\downarrow & & {\scriptstyle \subseteq}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ I & \xrightarrow[\cong]{} & \{y\} \times I & \xrightarrow[\subseteq]{} & Y \times I & \xrightarrow[H]{} & B \end{array}$$

for each $y \in Y$.

In the following exercise, we verify existence of lifts. The argument is similar to the proof in lecture of the homotopy lifting property for covering maps.

Exercise 9. Let $p : E \longrightarrow B$ be a covering map and consider any solid commutative diagram of the form

(8)
$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\;g\;} & E \\ {\scriptstyle \subseteq}\downarrow & \overset{\xi}{\underset{\exists}{\nearrow}} & \downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow[H]{} & B \end{array}$$

in Top. The purpose of this exercise is to prove that a lift ξ exists.

(a) Suppose for each $y \in Y$ there exists a neighborhood $N_y \subseteq Y$ of y and a continuous map ξ_y which makes the diagram

(9)
$$\begin{array}{ccccc} N_y \times \{0\} & \xrightarrow{\;\subseteq\;} & Y \times \{0\} & \xrightarrow{\;g\;} & E \\ {\scriptstyle \subseteq}\downarrow & & \overset{\xi_y}{\underset{\exists}{\nearrow}} & & \downarrow{\scriptstyle p} \\ N_y \times I & \xrightarrow[\subseteq]{} & Y \times I & \xrightarrow[H]{} & B \end{array}$$

in Top commute. Use unique path lifting to prove that the maps ξ_y determine a well-defined continuous map ξ which makes the diagram (8) commute.

(b) Let $y \in Y$. Prove there exists an integer $n > 0$ and a neighborhood $N_y \subseteq Y$ of y such that for each $0 \le i < n$ there exists an open subset $U_\alpha \subseteq B$ evenly covered by p which satisfies

$$H\big(N_y \times [i/n, (i+1)/n]\big) \subseteq U_\alpha.$$

(c) Use part (b) to construct a map $\xi_{y,0}$ which makes the diagram

$$\begin{array}{ccccc} N_y \times \{0\} & \xrightarrow{\;\subseteq\;} & Y \times \{0\} & \xrightarrow{\;g\;} & E \\ {\scriptstyle \subseteq}\downarrow & & \overset{\xi_{y,0}}{\nearrow} & & \downarrow{\scriptstyle p} \\ N_y \times [0/n, 1/n] & \xrightarrow[\subseteq]{} & Y \times I & \xrightarrow[H]{} & B \end{array}$$

commute.

(d) Use part (b) to construct a map $\xi_{y,1}$ which makes the diagram

$$\begin{array}{ccccc} & & & & E \\ & & \overset{\xi_{y,1}}{\nearrow} & & \downarrow{\scriptstyle p} \\ N_y \times [1/n, 2/n] & \xrightarrow[\subseteq]{} & Y \times I & \xrightarrow[H]{} & B \end{array}$$

commute and such that $\xi_{y,1}(-, 1/n) = \xi_{y,0}(-, 1/n)$.

(e) Similar to part (d), use an induction argument to construct maps $\xi_{y,i}$ which make the diagram

$$\begin{array}{ccccc} & & & & E \\ & & & \overset{\xi_{y,i}}{\nearrow} & \downarrow{\scriptstyle p} \\ N_y \times [i/n, (i+1)/n] & \xrightarrow[\subseteq]{} & Y \times I & \xrightarrow[H]{} & B \end{array}$$

commute and such that $\xi_{y,i}(-, i/n) = \xi_{y,i-1}(-, i/n)$.

(f) Conclude that the maps $\xi_{y,i}$ determine a well-defined continuous map ξ_y which makes the diagram (9) commute.

(g) Note that together with Exercise 8, this completes the proof of Theorem 7.

Recall the following from point-set topology.

Theorem 10. *Let X, Y, Z be spaces.*

(a) *If Y is Hausdorff and locally compact, then there are isomorphisms*

$$\hom_{\mathsf{Top}}(X \times Y, Z) \cong \hom_{\mathsf{Top}}(X, \operatorname{Map}(Y, Z))$$

natural in such X, Y, Z.

(b) *If X and Y are Hausdorff and Y is locally compact, then there are homeomorphisms*

$$\operatorname{Map}(X \times Y, Z) \cong \operatorname{Map}(X, \operatorname{Map}(Y, Z))$$
$$Z^{X \times Y} \cong (Z^Y)^X \qquad \textit{(exponential notation)}$$

natural in such X, Y, Z.

Here, the mapping space $B^A := \operatorname{Map}(A, B)$ is the set of all continuous maps $A \longrightarrow B$, equipped with the compact-open topology.

Exercise 11. Let A, X, Y be locally compact Hausdorff spaces and $i : A \longrightarrow X$ a cofibration. Suppose Z is a space and consider any solid commutative diagram of the form

(10)
$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{g} & \operatorname{Map}(X, Z) \\ {\scriptstyle \subseteq}\downarrow & \overset{\xi}{\nearrow}\ \exists & \downarrow{\scriptstyle (i,\mathrm{id}) =: p} \\ Y \times I & \xrightarrow[H]{} & \operatorname{Map}(A, Z) \end{array}$$

in Top. The purpose of this exercise is to prove that a lift ξ exists.

(a) Use the naturality of the isomorphisms in Theorem 10 to prove that the lifting problem in (10) is equivalent to the lifting problem

(11)
$$\begin{array}{ccc} A \times Y & \xrightarrow{H} & \operatorname{Map}(I, Z) \\ {\scriptstyle i \times \mathrm{id}}\downarrow & \overset{\xi}{\nearrow}\ \exists & \downarrow{\scriptstyle \mathrm{ev}_0} \\ X \times Y & \xrightarrow[g]{} & Z \end{array}$$

in Top.

(b) Use the naturality of the isomorphisms in Theorem 10 to prove that the lifting problem in (11) is equivalent to the lifting problem

$$\begin{array}{ccccc} A & \xrightarrow{H} & \mathrm{Map}(Y, \mathrm{Map}(I, Z)) & \xrightarrow{\cong} & \mathrm{Map}(I, \mathrm{Map}(Y, Z)) \\ {\scriptstyle i}\downarrow & & \overset{\exists\, \xi}{\nearrow} & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow[g]{} & \mathrm{Map}(Y, Z) & = & \mathrm{Map}(Y, Z) \end{array} \tag{12}$$

in Top.

(c) Conclude from [Series 9 & 10] that (12) has a lift ξ, and hence (10) has a lift ξ.

(d) Let $n \geq 1$. Conclude from [Series 9 & 10] that the induced map $p := (i, \mathrm{id}) : \mathrm{Map}(D^n, Z) \longrightarrow \mathrm{Map}(\partial D^n, Z)$ has the homotopy lifting property with respect to Y.

(e) Conclude that $p : \mathrm{Map}(I, Z) \longrightarrow Z \times Z$ has the homotopy lifting property with respect to Y (by taking $n = 1$).

Here are some references for this material: [1, Chapter 1], [2, Chapter 11], [3, Chapter 10], [4, Section 2.2].

References

[1] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic topology from a homotopical viewpoint.* Universitext. Springer-Verlag, New York, 2002. Translated from the Spanish by Stephen Bruce Sontz.

[2] Brayton Gray. *Homotopy theory.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. An introduction to algebraic topology, Pure and Applied Mathematics, Vol. 64.

[3] Joseph J. Rotman. *An introduction to algebraic topology*, volume 119 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1988.

[4] Edwin H. Spanier. *Algebraic topology.* Springer-Verlag, New York, 1981. Corrected reprint.