Recall the following from point-set topology.

**Theorem 1.** Let $X, Y, Z$ be spaces.

(a) If $Y$ is Hausdorff and locally compact, then there are isomorphisms

$$\text{hom}_{\text{Top}}(X \times Y, Z) \cong \text{hom}_{\text{Top}}(X, \text{Map}(Y, Z))$$

natural in such $X, Y, Z$.

(b) If $X$ and $Y$ are Hausdorff and $Y$ is locally compact, then there are homeomorphisms

$$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$$

$$Z^{X \times Y} \cong (Z^Y)^X \quad \text{(exponential notation)}$$

natural in such $X, Y, Z$.

Here, the mapping space $B^A := \text{Map}(A, B)$ is the set of all continuous maps $A \to B$, equipped with the compact-open topology.

**Exercise 2.** Let $X$ be a locally compact Hausdorff space and $p : E \to B$ a fibration. Let $Y$ be a space and consider any solid commutative diagram of the form

(1) \[ Y \times \{0\} \xrightarrow{g} \text{Map}(X, E) \]

\[ \subseteq \xrightarrow{\xi} \exists \xrightarrow{(\text{id}, p) = q} \]

\[ Y \times I \xrightarrow{H} \text{Map}(X, B) \]

in $\text{Top}$. The purpose of this exercise is to prove that a lift $\xi$ exists.

(a) Use the naturality of the isomorphisms in Theorem 1 to prove that the lifting problem in (1) is equivalent to the lifting problem

(2) \[ Y \times X \times \{0\} \xrightarrow{g} E \]

\[ \subseteq \xrightarrow{\xi} \exists \xrightarrow{p} \]

\[ Y \times X \times I \xrightarrow{H} B \]

in $\text{Top}$.

(b) Conclude that (2) has a lift $\xi$.

(c) Conclude that the induced map

$$q := (\text{id}, p) : \text{Map}(X, E) \to \text{Map}(X, B)$$

is a fibration.
**Proposition 3.** Let $A, B$ be spaces. If $A' \subseteq A$ and $B' \subseteq B$ are closed subspaces, then $A' \times B' \subseteq A \times B$ is a closed subspace.

**Exercise 4.** Let $B, Y$ be spaces. Suppose $B_1, B_2 \subseteq B$ are closed subspaces such that $B = B_1 \cup B_2$. Consider the left-hand side diagram of inclusions

$$
\begin{array}{ccc}
B_1 \cap B_2 & \subseteq & B_2 \\
\subseteq & & \subseteq \\
B_1 & \subseteq & B
\end{array}
\quad
\begin{array}{ccc}
(\ B_1 \cap B_2\ ) \times Y & \subseteq & B_2 \times Y \\
\subseteq & & \subseteq \\
B_1 \times Y & \subseteq & B \times Y
\end{array}
$$

in $\mathsf{Top}$. Apply $- \times Y$ to obtain the right-hand side diagram of inclusions.

(a) Prove that the left-hand diagram is a pushout diagram in $\mathsf{Top}$ by verifying the universal property of pushouts.

(b) Prove that the right-hand diagram is a pushout diagram in $\mathsf{Top}$ by using Proposition 3 and verifying the universal property of pushouts.

(c) Conclude that $- \times Y$ preserves pushout diagrams of closed inclusions.

Recall the following from [Series 9 & 10].

**Proposition 5.** Let $X$ be a space and $A \subseteq X$ a closed subspace. Then the inclusion map $i : A \rightarrow X$ is a cofibration if and only if there exists a continuous map $r$ which makes the diagram

$$
\begin{array}{ccc}
X \times \{0\} \cup A \times I & \xrightarrow{id} & X \times \{0\} \cup A \times I \\
\subseteq & & \subseteq \\
X \times I & \xrightarrow{r} & X \times I
\end{array}
$$

commute; i.e., if and only if the subspace $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

**Exercise 6.** Let $X$ be a space and $A \subseteq X$ a closed subspace. Suppose the inclusion map $i : A \rightarrow X$ is a cofibration. Let $Y$ be a space. The purpose of this exercise is to prove that $i \times \text{id} : A \times Y \rightarrow X \times Y$ is a cofibration.

(a) Use Exercise 4 to prove that the diagram of inclusions

$$
\begin{array}{ccc}
A \times \{0\} & \subseteq & A \times I \\
\subseteq & & \subseteq \\
X \times \{0\} & \subseteq & X \times \{0\} \cup A \times I
\end{array}
$$

is a pushout diagram in $\mathsf{Top}$. 

(b) Apply \(- \times Y\) to diagram (3) and use Exercise 4 to prove that the resulting diagram of inclusions

\[
\begin{array}{ccc}
A \times \{0\} \times Y & \subseteq & A \times I \times Y \\
\subseteq & & \subseteq \\
X \times \{0\} \times Y & \subseteq & (X \times \{0\} \cup A \times I) \times Y
\end{array}
\]

is a pushout diagram in \(\text{Top}\).

(c) Conclude the following:

\[
(X \times \{0\} \cup A \times I) \times Y \cong (X \times \{0\} \times Y) \cup (A \times I \times Y) \\
\cong (X \times Y \times \{0\}) \cup (A \times Y \times I)
\]

(d) Use Proposition 5 to prove that \(i \times \text{id} : A \times Y \longrightarrow X \times Y\) is a cofibration.

Recall from Exercise 11 in [Series 11] that if \(i : A \longrightarrow X\) is a cofibration between locally compact Hausdorff spaces and \(Z\) is any space, then the induced map

\[
p := (i, \text{id}) : \text{Map}(X, Z) \longrightarrow \text{Map}(A, Z)
\]

has the homotopy lifting property with respect to every locally compact Hausdorff space \(Y\). The purpose of Exercise 8 is to prove that \(p\) has the homotopy lifting property with respect to any space \(Y\), and hence \(p\) is a fibration, provided that \(i : A \longrightarrow X\) is furthermore a closed inclusion.

**Remark 7.** It can be shown that every cofibration \(i : A \longrightarrow X\) between Hausdorff spaces is isomorphic to a closed inclusion. Hence, Exercise 8 proves that every cofibration \(i : A \longrightarrow X\) between locally compact Hausdorff spaces induces a fibration \(\text{Map}(X, Z) \longrightarrow \text{Map}(A, Z)\) on mapping spaces, for any space \(Z\).

**Exercise 8.** Let \(A, X\) be locally compact Hausdorff spaces and \(A \subseteq X\) a closed subspace. Suppose the inclusion map \(i : A \longrightarrow X\) is a cofibration. Let \(Y, Z\) be spaces and consider any solid commutative diagram of the form

\[
\begin{array}{ccc}
Y \times \{0\} & \xrightarrow{g} & \text{Map}(X, Z) \\
\subseteq & \xymatrix{\wr} & \xymatrix{\exists} \\
Y \times I & \xrightarrow{H} & \text{Map}(A, Z)
\end{array}
\]

in \(\text{Top}\). The purpose of this exercise is to prove that a lift \(\xi\) exists.
(a) Use the naturality of the isomorphisms in Theorem 1 to prove that the lifting problem in (5) is equivalent to the lifting problem

\[
\begin{array}{ccc}
A \times Y & \xrightarrow{H} & \text{Map}(I, Z) \\
\downarrow \scriptstyle{i \times \text{id}} & \searrow \scriptstyle{\xi} & \downarrow \scriptstyle{\text{ev}_0} \\
X \times Y & \xrightarrow{g} & Z
\end{array}
\]

in Top.

(b) Use Exercise 6 to prove that (6) has a lift \( \xi \).

(c) Conclude that the induced map

\[ p := (i, \text{id}) : \text{Map}(X, Z) \longrightarrow \text{Map}(A, Z) \]

is a fibration.

(d) Let \( n \geq 1 \). Recall from [Series 9 & 10] that the inclusion \( i : \partial D^n \longrightarrow D^n \) is a cofibration. Conclude that the induced map

\[ p := (i, \text{id}) : \text{Map}(D^n, Z) \longrightarrow \text{Map}(\partial D^n, Z) \]

is a fibration.

(e) By taking \( n = 1 \), conclude that \( p : \text{Map}(I, Z) \longrightarrow Z \times Z \) is a fibration.

Here are some references for this material: [1, Chapter 4], [2, Sections 1.4, 2.8].

REFERENCES
