Elements of Homotopy **Fall 2008**

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Series 12

Recall the following from point-set topology.

Theorem 1. *Let X, Y, Z be spaces.*

(a) *If Y is Hausdorff and locally compact, then there are isomorphisms*

$$\hom_{\mathsf{Top}}(X \times Y, Z) \cong \hom_{\mathsf{Top}}(X, \mathrm{Map}(Y, Z))$$

natural in such X, Y, Z.

(b) *If X and Y are Hausdorff and Y is locally compact, then there are homeomorphisms*

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$
$$Z^{X \times Y} \cong (Z^Y)^X \qquad \textit{(exponential notation)}$$

natural in such X, Y, Z.

Here, the mapping space $B^A := \mathrm{Map}(A, B)$ is the set of all continuous maps $A \longrightarrow B$, equipped with the compact-open topology.

Exercise 2. Let X be a locally compact Hausdorff space and $p : E \longrightarrow B$ a fibration. Let Y be a space and consider any solid commutative diagram of the form

(1)
$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{g} & \mathrm{Map}(X, E) \\ {\scriptstyle\subseteq}\downarrow & \overset{\xi}{\nearrow}\ \exists & \downarrow {\scriptstyle(\mathrm{id},p)=:q} \\ Y \times I & \xrightarrow[H]{} & \mathrm{Map}(X, B) \end{array}$$

in Top. The purpose of this exercise is to prove that a lift ξ exists.

(a) Use the naturality of the isomorphisms in Theorem 1 to prove that the lifting problem in (1) is equivalent to the lifting problem

(2)
$$\begin{array}{ccc} Y \times X \times \{0\} & \xrightarrow{g} & E \\ {\scriptstyle\subseteq}\downarrow & \overset{\xi}{\nearrow}\ \exists & \downarrow {\scriptstyle p} \\ Y \times X \times I & \xrightarrow[H]{} & B \end{array}$$

in Top.

(b) Conclude that (2) has a lift ξ.

(c) Conclude that the induced map

$$q := (\mathrm{id}, p) : \mathrm{Map}(X, E) \longrightarrow \mathrm{Map}(X, B)$$

is a fibration.

Recall the following from point-set topology.

Proposition 3. *Let A, B be spaces. If $A' \subseteq A$ and $B' \subseteq B$ are closed subspaces, then $A' \times B' \subseteq A \times B$ is a closed subspace.*

Exercise 4. Let B, Y be spaces. Suppose $B_1, B_2 \subseteq B$ are closed subspaces such that $B = B_1 \cup B_2$. Consider the left-hand side diagram of inclusions

$$
\begin{array}{ccc}
B_1 \cap B_2 & \xrightarrow{\subseteq} & B_2 \\
{\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\
B_1 & \xrightarrow[\subseteq]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
(B_1 \cap B_2) \times Y & \xrightarrow{\subseteq} & B_2 \times Y \\
{\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\
B_1 \times Y & \xrightarrow[\subseteq]{} & B \times Y
\end{array}
$$

in Top. Apply $- \times Y$ to obtain the right-hand side diagram of inclusions.

(a) Prove that the left-hand diagram is a pushout diagram in Top by verifying the universal property of pushouts.
(b) Prove that the right-hand diagram is a pushout diagram in Top by using Proposition 3 and verifying the universal property of pushouts.
(c) Conclude that $- \times Y$ preserves pushout diagrams of closed inclusions.

Recall the following from [Series 9 & 10].

Proposition 5. *Let X be a space and $A \subseteq X$ a closed subspace. Then the inclusion map $i : A \longrightarrow X$ is a cofibration if and only if there exists a continuous map r which makes the diagram*

$$
\begin{array}{ccc}
X \times \{0\} \cup A \times I & \xrightarrow{\mathrm{id}} & X \times \{0\} \cup A \times I \\
{\scriptstyle\subseteq}\downarrow & \overset{r}{\underset{\exists}{\nearrow}} & \\
X \times I & &
\end{array}
$$

commute; i.e., if and only if the subspace $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Exercise 6. Let X be a space and $A \subseteq X$ a closed subspace. Suppose the inclusion map $i : A \longrightarrow X$ is a cofibration. Let Y be a space. The purpose of this exercise is to prove that $i \times \mathrm{id} : A \times Y \longrightarrow X \times Y$ is a cofibration.

(a) Use Exercise 4 to prove that the diagram of inclusions

$$
\begin{array}{ccc}
A \times \{0\} & \xrightarrow{\subseteq} & A \times I \\
{\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\
X \times \{0\} & \xrightarrow[\subseteq]{} & X \times \{0\} \cup A \times I
\end{array}
\tag{3}
$$

is a pushout diagram in Top.

(b) Apply $- \times Y$ to diagram (3) and use Exercise 4 to prove that the resulting diagram of inclusions

(4)
$$\begin{CD}
A \times \{0\} \times Y @>{\subseteq}>> A \times I \times Y \\
@V{\subseteq}VV @VV{\subseteq}V \\
X \times \{0\} \times Y @>>{\subseteq}> \big(X \times \{0\} \cup A \times I\big) \times Y
\end{CD}$$

is a pushout diagram in Top.

(c) Conclude the following:

$$\begin{aligned}
\big(X \times \{0\} \cup A \times I\big) \times Y &\cong \big(X \times \{0\} \times Y\big) \cup \big(A \times I \times Y\big) \\
&\cong \big(X \times Y \times \{0\}\big) \cup \big(A \times Y \times I\big)
\end{aligned}$$

(d) Use Proposition 5 to prove that $i \times \mathrm{id} : A \times Y \longrightarrow X \times Y$ is a cofibration.

Recall from Exercise 11 in [Series 11] that if $i : A \longrightarrow X$ is a cofibration between locally compact Hausdorff spaces and Z is any space, then the induced map

$$p := (i, \mathrm{id}) : \mathrm{Map}(X, Z) \longrightarrow \mathrm{Map}(A, Z)$$

has the homotopy lifting property with respect to every locally compact Hausdorff space Y. The purpose of Exercise 8 is to prove that p has the homotopy lifting property with respect to any space Y, and hence p is a fibration, provided that $i : A \longrightarrow X$ is furthermore a closed inclusion.

Remark 7. It can be shown that every cofibration $i : A \longrightarrow X$ between Hausdorff spaces is isomorphic to a closed inclusion. Hence, Exercise 8 proves that every cofibration $i : A \longrightarrow X$ between locally compact Hausdorff spaces induces a fibration $\mathrm{Map}(X, Z) \longrightarrow \mathrm{Map}(A, Z)$ on mapping spaces, for any space Z.

Exercise 8. Let A, X be locally compact Hausdorff spaces and $A \subseteq X$ a closed subspace. Suppose the inclusion map $i : A \longrightarrow X$ is a cofibration. Let Y, Z be spaces and consider any solid commutative diagram of the form

(5)
$$\begin{CD}
Y \times \{0\} @>{g}>> \mathrm{Map}(X, Z) \\
@V{\subseteq}VV @VV{(i,\mathrm{id}) =: p}V \\
Y \times I @>>{H}> \mathrm{Map}(A, Z)
\end{CD}
\qquad (\exists\, \xi : Y \times I \dashrightarrow \mathrm{Map}(X, Z))$$

in Top. The purpose of this exercise is to prove that a lift ξ exists.

(a) Use the naturality of the isomorphisms in Theorem 1 to prove that the lifting problem in (5) is equivalent to the lifting problem

$$
\begin{array}{ccc}
A \times Y & \xrightarrow{H} & \mathrm{Map}(I, Z) \\
{\scriptstyle i\times \mathrm{id}}\big\downarrow & \overset{\xi}{\nearrow}\ \exists & \big\downarrow{\scriptstyle \mathrm{ev}_0} \\
X \times Y & \xrightarrow[g]{} & Z
\end{array} \tag{6}
$$

in Top.

(b) Use Exercise 6 to prove that (6) has a lift ξ.

(c) Conclude that the induced map

$$p := (i, \mathrm{id}) : \mathrm{Map}(X, Z) \longrightarrow \mathrm{Map}(A, Z)$$

is a fibration.

(d) Let $n \geq 1$. Recall from [Series 9 & 10] that the inclusion $i : \partial D^n \longrightarrow D^n$ is a cofibration. Conclude that the induced map

$$p := (i, \mathrm{id}) : \mathrm{Map}(D^n, Z) \longrightarrow \mathrm{Map}(\partial D^n, Z)$$

is a fibration.

(e) By taking $n = 1$, conclude that $p : \mathrm{Map}(I, Z) \longrightarrow Z \times Z$ is a fibration.

Here are some references for this material: [1, Chapter 4], [2, Sections 1.4, 2.8].

References

[1] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic topology from a homotopical viewpoint*. Universitext. Springer-Verlag, New York, 2002. Translated from the Spanish by Stephen Bruce Sontz.

[2] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York, 1981. Corrected reprint.