Elements of Homotopy **Fall 2008**

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Series 13

Let B be a path connected based space and recall that if $p : E \longrightarrow B$ is a fibration (or locally trivial bundle) with nonempty fiber F, then the sequence $F \longrightarrow E \longrightarrow B$ fits into a long exact sequence of the form

$$\cdots \longrightarrow \pi_k(F) \longrightarrow \pi_k(E) \longrightarrow \pi_k(B) \longrightarrow \pi_{k-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow *.$$

Exercise 1. Let p be prime to q. Recall the following fibrations (or locally trivial bundles) $p : E \longrightarrow B$ with fiber F, which we display in the form $F \longrightarrow E \longrightarrow B$:

$$\begin{aligned} \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1&, \\ \mathbb{Z}^2 \longrightarrow \mathbb{R}^2 \longrightarrow T&, \\ \mathbb{Z}/2\mathbb{Z} \longrightarrow S^n \longrightarrow \mathbb{R}P^n&, \qquad (n \geq 1), \\ S^1 \longrightarrow S^3 \longrightarrow S^2&, \qquad (n = 1), \\ S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n&, \qquad (n \geq 1), \\ \mathbb{Z}/p\mathbb{Z} \longrightarrow S^3 \longrightarrow L(p,q)&. \end{aligned}$$

Use the associated long exact sequence above together with earlier results to prove the following:

(a) $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_k(S^1) = 0$ for $k \neq 1$.

(b) $\pi_1(T) \cong \mathbb{Z}^2$ and $\pi_k(T) = 0$ for $k \neq 1$.

(c) If $n \geq 2$, then $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_k(\mathbb{R}P^n) \cong \pi_k(S^n)$ for $k \neq 1$.

(d) $\pi_k(S^3) \cong \pi_k(S^2)$ for $k \geq 3$.

(e) There is a short exact sequence

$$0 \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2) \longrightarrow \mathbb{Z} \longrightarrow 0$$

of abelian groups.

(f) $\pi_k(\mathbb{C}P^n) \cong \pi_k(S^{2n+1})$ for $k \geq 3$.

(g) There is a short exact sequence

$$0 \longrightarrow \pi_2(S^{2n+1}) \longrightarrow \pi_2(\mathbb{C}P^n) \longrightarrow \mathbb{Z} \longrightarrow 0$$

of abelian groups.

(h) $\pi_1\big(L(p,q)\big) \cong \mathbb{Z}/p\mathbb{Z}$ and $\pi_k\big(L(p,q)\big) \cong \pi_k(S^3)$ for $k \neq 1$.

Exercise 2. Let B, E, X be spaces. Prove the following:

(a) The map $p : \emptyset \longrightarrow B$ is a fibration.

(b) The map $p : E \longrightarrow *$ is a fibration.

(c) The map $i : \emptyset \longrightarrow X$ is a cofibration.

Note that the empty space $\emptyset$ has no path components; i.e. $\pi_0(\emptyset) = \emptyset$.

Exercise 3. Let $p : E \longrightarrow B$ be a fibration. Prove the following:

(a) If $A \subseteq E$ is a path component of E, then $p(A) \subseteq B$ is a path component of B.
(b) If B is path connected and E is nonempty, then p is a surjective map.

Exercise 4. Let $p : E \longrightarrow B$ be a fibration, $b_0 \in B$, and $F := p^{-1}(b_0) \subseteq E$ the fiber over b_0. Assume F is nonempty. Denote by $i : F \longrightarrow E$ the inclusion map. Prove the following:

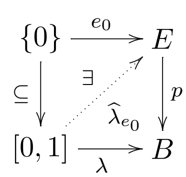
(a) If B is path connected, then the induced map $\pi_0(i) : \pi_0(F) \longrightarrow \pi_0(E)$ on path components is a surjection.

The following will be useful for studying the first several maps in the long exact sequence associated to a fibration (or locally trivial bundle).

Exercise 5. Let $p : E \longrightarrow B$ be a fibration, $b_0 \in B$, and $F := p^{-1}(b_0) \subseteq E$ the fiber over b_0. Assume F is nonempty. A right action of $\pi_1(B, b_0)$ on the set $\pi_0(F)$ is defined by lifting loops as follows:

$$\pi_0(F) \times \pi_1(B, b_0) \longrightarrow \pi_0(F), \quad ([e_0], [\lambda]) \longmapsto [e_0][\lambda] := [\widehat{\lambda}_{e_0}(1)]. \tag{1}$$

Here, $\widehat{\lambda}_{e_0} : [0,1] \longrightarrow E$ is a lift of the loop $\lambda : [0,1] \longrightarrow B$ such that $\widehat{\lambda}_{e_0}(0) = e_0$; in other words, $\widehat{\lambda}_{e_0}$ is a map which makes the diagram

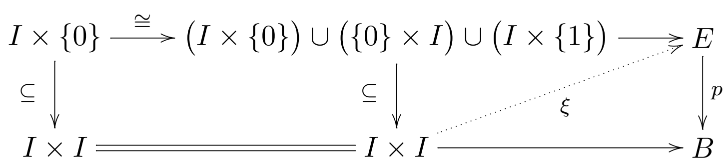


commute.

(a) Prove that (1) is a well-defined function.
(b) Prove that (1) defines a right $\pi_1(B, b_0)$-action on the set $\pi_0(F)$ of path components of F.
(c) If B is path connected, prove that the induced map $\pi_0(F) \longrightarrow \pi_0(E)$ is surjective and partitions the set $\pi_0(F)$ into $\pi_1(B, b_0)$-orbits; conclude that $\pi_0(E)$ is the orbit space $\pi_0(F)/\pi_1(B, b_0)$.
(d) Prove that the isotropy subgroup of $[e_0] \in \pi_0(F)$ is the image of

$$\pi_1(p) : \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0).$$

Consider solid commutative diagrams of the form



in Top, and use existence of a lift ξ.

Here are some references for this material: [1, Chapter 11], [2, Chapter 9], [3, Chapter 2.3], [4, Section 3.2].

References

[1] Brayton Gray. *Homotopy theory*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. An introduction to algebraic topology, Pure and Applied Mathematics, Vol. 64.

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[3] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York, 1981. Corrected reprint.

[4] Tammo tom Dieck. *Algebraic topology*. European Mathematical Society, Zurich, 2008.