Elements of Homotopy **Fall 2008**
Prof. Kathryn Hess

Series 14

Let $p : E \longrightarrow B$ be a continuous map and consider the pullback square

$$\begin{tikzcd} E^I \arrow[rrd, bend left, "\mathrm{ev}_0"] \arrow[rd, dotted, "r", "\exists!"'] \arrow[rdd, bend right, "p^I"'] & & \\ & E \times_B B^I \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ & B^I \arrow[r, "\mathrm{ev}_0"'] & B \end{tikzcd}$$

in Top. Since the outer diagram commutes, there exists a unique map r which makes the diagram commute. Recall that a *lifting function* for p is a right inverse Γ in Top of the induced map r; i.e., a continuous map $\Gamma : E \times_B B^I \longrightarrow E^I$ such that $r\Gamma = \mathrm{id}$. The following was proved in lecture.

Theorem 1. *A map $p : E \longrightarrow B$ is a fibration if and only if p has a lifting function.*

Recall that associated to each lifting function Γ is the *holonomy* map γ defined as

$$\begin{tikzcd} E \times_B B^I \arrow[r, "\Gamma"] & E^I \arrow[r, "\mathrm{ev}_1"] & E \\ F \times \Omega B \arrow[u, "\subseteq"] \arrow[rr, "\gamma"] & & F. \arrow[u, "\subseteq"'] \end{tikzcd}$$

The holonomy map determines the connecting homomorphism in the long exact sequence associated to a fibration. The purpose of the following exercise is to calculate the holonomy map for two particular fibrations.

Exercise 2. Let Y be a based space with basepoint $y_0 \in Y$, and recall that the fibrations $p : PY \longrightarrow Y$ and $p : \mathcal{L}Y \longrightarrow Y$ are defined by the pullback diagrams

$$\begin{tikzcd} PY \arrow[r] \arrow[d, "p"'] & Y^I \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ Y \cong * \times Y \arrow[r, "(y_0,\mathrm{id})"'] & Y \times Y \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}Y \arrow[r] \arrow[d, "p"'] & Y^I \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ Y \arrow[r, "(\mathrm{id},\mathrm{id})"'] & Y \times Y \end{tikzcd}$$

in Top. It follows that PY and $\mathcal{L}Y$ are isomorphic to the subspaces

$$PY \cong \{\omega \in Y^I \mid \omega(0) = y_0\} \subseteq Y^I,$$
$$\mathcal{L}Y \cong \{\omega \in Y^I \mid \omega(0) = \omega(1)\} \subseteq Y^I,$$

of Y^I. Under these isomorphisms, each fibration satisfies $p = \mathrm{ev}_1$.

(a) Calculate the holonomy of the fibration $p : PY \longrightarrow Y$.
(b) Calculate the holonomy of the fibration $p : \mathcal{L}Y \longrightarrow Y$.

A continuous map $g : X \longrightarrow X'$ is a *weak equivalence* if the induced map $\pi_0(g) : \pi_0(X) \longrightarrow \pi_0(X')$ on path components is a bijection and the induced map $\pi_n(g) : \pi_n(X, x) \longrightarrow \pi_n(X', g(x))$ on homotopy groups is an isomorphism for every $n \geq 1$ and $x \in X$.

Exercise 3. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ be fibrations with path connected base B. Let $b_0 \in B$, $F := p^{-1}(b_0) \subseteq E$, and $F' := p'^{-1}(b_0) \subseteq E'$. Assume the fibers F, F' are nonempty. Consider any commutative diagram of the form

$$\begin{array}{ccccc} F & \xrightarrow{\subseteq} & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| \\ F' & \xrightarrow{\subseteq} & E' & \xrightarrow{p'} & B \end{array}$$

in Top. Prove the following:

(a) $\pi_0(f)$ is a map of $\pi_1(B, b_0)$-sets; i.e., verify that

$$\pi_0(f)\big([e_0][\lambda]\big) = \Big(\pi_0(f)([e_0])\Big)[\lambda]$$

for every $[e_0] \in \pi_0(F)$ and $[\lambda] \in \pi_1(B, b_0)$.

(b) If g is a weak equivalence then f is a weak equivalence.
(c) If f is a weak equivalence then g is a weak equivalence.

Use the associated long exact sequence

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(E) & \longrightarrow & \pi_1(B) & \xrightarrow{\partial} & \pi_0(F) & \longrightarrow & \pi_0(E) & \longrightarrow * \\ & \downarrow{\scriptstyle \pi_1(f)} & & \downarrow{\scriptstyle \pi_1(g)} & & \| & & \downarrow{\scriptstyle \pi_0(f)} & & \downarrow{\scriptstyle \pi_0(g)} & \quad \| \\ \cdots \longrightarrow & \pi_1(F') & \longrightarrow & \pi_1(E') & \longrightarrow & \pi_1(B) & \xrightarrow{\partial} & \pi_0(F') & \longrightarrow & \pi_0(E') & \longrightarrow * \end{array}$$

and the fact that the connecting map $\partial : \pi_1(B) \longrightarrow \pi_0(F)$ satisfies $\partial([\lambda]) = [e_0][\lambda]$.

Using similar arguments, the previous exercise generalizes to the following.

Theorem 4. *Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ be fibrations with path connected base B and B', respectively. Let $b_0 \in B$, $F := p^{-1}(b_0) \subseteq E$, and $F' := p'^{-1}\big(h(b_0)\big) \subseteq E'$. Assume the fibers F, F' are nonempty. Consider any commutative diagram of the form*

$$\begin{array}{ccccc} F & \xrightarrow{\subseteq} & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ F' & \xrightarrow{\subseteq} & E' & \xrightarrow{p'} & B' \end{array}$$

in Top. *Suppose h is a weak equivalence. Then f is a weak equivalence if and only if g is a weak equivalence.*

Exercise 5. Let $p : E \longrightarrow B$ be a fibration with path connected base B. Let $b_0 \in B$ and consider the fiber $F := p^{-1}(b_0) \subseteq E$. Prove the following.

(a) If F is contractible, then $p : E \longrightarrow B$ is a weak equivalence.
(b) If F and B are contractible, then E is contractible.

Use the fact that a space X is contractible if and only if the map $X \longrightarrow *$ is a weak equivalence.

Here are some references for this material: [1, Section 3.2]

References

[1] tom Dieck Tammo. *Algebraic topology.* European Mathematical Society, Zurich, 2008.