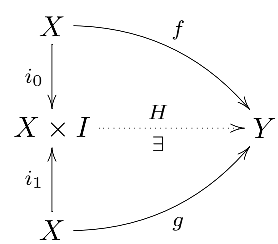
Elements of Homotopy **Fall 2008**

Prof. Kathryn Hess

Series 1 & 2

Recall that if $f, g : X \longrightarrow Y$ are continuous functions, we say that f is *homotopic* to g, denoted by $f \simeq g$, if there exists a continuous function $H : X \times I \longrightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each $x \in X$. Such a function H is called a *homotopy* from f to g. Hence, in terms of diagrams, f is homotopic to g if and only if there exists a continuous function H which makes the diagram



commute. Here, $i_0(x) := (x, 0)$ and $i_1(x) := (x, 1)$ for each $x \in X$. We will refer to this diagram in a later exercise. Sometimes we use the term *space* to denote a topological space. Sometimes we use the term *map of spaces*, or simply *map*, to denote a continuous function.

Exercise 1. Let X, Y be topological spaces. Prove that the homotopy relation $\simeq$ is an equivalence relation on the set of all continuous functions $f : X \longrightarrow Y$.

Hence the set of all maps $f : X \longrightarrow Y$ is partitioned into equivalence classes via $\simeq$. The equivalence classes are called *homotopy classes*, and the set of all homotopy classes is denoted by $[X, Y]$. If $f : X \longrightarrow Y$ is a map, then the homotopy class of f is denoted by $[f]$.

Exercise 2. Let X, Y be based topological spaces. Prove that the based homotopy relation $\simeq_*$ is an equivalence relation on the set of all based continuous functions $X \longrightarrow Y$.

Recall that a *based homotopy* is a continuous function $H : X \times I \longrightarrow Y$ such that $H(*, t) = *$ for each $t \in I$. Here we use the notation $*$ to denote the basepoint of X and the basepoint of Y. The set of all based maps $f : X \longrightarrow Y$ is partitioned into equivalence classes via $\simeq_*$. The equivalence classes are called *based homotopy classes*, and the set of all based homotopy classes is denoted by $[X, Y]_*$. If $f : X \longrightarrow Y$ is a based map, then the homotopy class of f is denoted by $[f]_*$.

Exercise 3. Prove that composites of homotopic maps are homotopic. In other words, if $f, g : X \longrightarrow Y$ and $f', g' : Y \longrightarrow Z$ are maps such that $f \simeq g$ and $f' \simeq g'$, verify that $f' \circ f \simeq g' \circ g$.

Exercise 4. Prove that composites of based homotopic maps are based homotopic. In other words, if $f, g : X \longrightarrow Y$ and $f', g' : Y \longrightarrow Z$ are based maps such that $f \simeq_* g$ and $f' \simeq_* g'$, verify that $f' \circ f \simeq_* g' \circ g$.

A map $f : X \longrightarrow Y$ is a *homotopy equivalence* if there exists a map $g : Y \longrightarrow X$ such that $g \circ f \simeq \mathrm{id}$ and $f \circ g \simeq \mathrm{id}$. Two spaces X and Y are *homotopy equivalent* if there exists a homotopy equivalence $f : X \longrightarrow Y$.

A topological space X is *contractible* if the identity map $\mathrm{id} : X \longrightarrow X$ is homotopic to the constant map $c_{x_0} : X \longrightarrow X$ of some point $x_0 \in X$. Here, $c_{x_0}(x) := x_0$ for each $x \in X$. In particular, any one-point space is contractible, but the empty space $\emptyset$ is not contractible.

Exercise 5. Let X be a space and let Y be a convex subset of $\mathbb{R}^n$. Suppose $f, g : X \longrightarrow Y$ are maps such that $f(x_0) = g(x_0)$ for some $x_0 \in X$. Use a straight-line homotopy to prove that $f \simeq_* g$. Use this to prove that $\mathbb{R}^n$ and I are contractible.

Exercise 6. Suppose Y is a contractible space. Prove that any two maps $f, g : X \longrightarrow Y$ are homotopic. Use this to prove that any two constant maps $c_{y_0} : Y \longrightarrow Y$ and $c_{y_1} : Y \longrightarrow Y$ are homotopic and that the identity map $\mathrm{id} : Y \longrightarrow Y$ is homotopic to any constant map $c_{y_0} : Y \longrightarrow Y$.

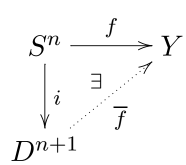
Exercise 7. Prove that a space is contractible if and only if it has the same homotopy type as a one-point space. Use this to prove that two contractible spaces have the same homotopy type, and that any map between contractible spaces is a homotopy equivalence.

Let $n \geq 0$. The *n-sphere* is defined by $S^n := \{x \in \mathbb{R}^{n+1} | \, \|x\| = 1\}$ regarded as a subspace of $\mathbb{R}^{n+1}$. In particular, S^0 is the two-point subspace $\{-1, 1\}$ of $\mathbb{R}$. The *n-disk* is defined by $D^n := \{x \in \mathbb{R}^n | \, \|x\| \leq 1\}$ regarded as a subspace of $\mathbb{R}^n$. Denote by $i : S^n \longrightarrow D^{n+1}$ the obvious inclusion map. The following is a useful relation between homotopy and the extension of certain maps.

Exercise 8. Let $f : S^n \longrightarrow Y$ be a map and consider any point $x_0 \in S^n$. Prove that the following are equivalent:

(a) f is homotopic to a constant map

(b) there exists a map $\overline{f}$ which makes the diagram



commute

(c) f is based homotopic to a constant map, with the basepoint of S^n defined to be x_0.

Exercise 9. Suppose Y is a contractible space. Use the above exercise to prove that any map $f : S^n \longrightarrow Y$ has a continuous extension over D^{n+1}.

Homotopy theory and algebraic topology involve mappings from topology to algebra, and category theory gives a useful language to express this. Here we record some basic terminology, following very closely the summary given in [2, Chapter 2].

A category C consists of a collection of objects, a set $\hom_{\mathsf{C}}(X, Y)$ of morphisms (also called maps) between any two objects, an identity morphism $\mathrm{id} \in \hom_{\mathsf{C}}(X, X)$ for each object X, and a composition pairing

$$\circ : \hom_{\mathsf{C}}(Y, Z) \times \hom_{\mathsf{C}}(X, Y) \longrightarrow \hom_{\mathsf{C}}(X, Z)$$
$$Y \xrightarrow{g} Z, X \xrightarrow{f} Y \quad \longmapsto \quad X \xrightarrow{g \circ f} Z$$

for each triple of objects X, Y, Z. Composition must be associative and identity morphisms must act as two-sided identities:

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \mathrm{id} \circ f = f, \quad f \circ \mathrm{id} = f$$

whenever the indicated composites are defined. For example, there is the category Set of sets and functions, the category Top of topological spaces and continuous functions, the category Top_* of based topological spaces and based continuous functions, the category Grp of groups and homomorphisms, the category Ab of abelian groups and homomorphisms, and so on.

A functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a map of categories. It assigns to each object X of C an object $F(X)$ of D and to each morphism $f : X \longrightarrow Y$ of C a morphism $F(f) : F(X) \longrightarrow F(Y)$ of D in such a way that it respects composition and identity:

$$F(\mathrm{id}) = \mathrm{id}, \quad F(g \circ f) = F(g) \circ F(f).$$

A morphism $f : X \longrightarrow Y$ in C is an isomorphism if there exists a $g : Y \longrightarrow X$ in C such that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. In particular, functors preserve isomorphisms. A category C has an opposite category C^{op} with the same objects as C and with $\hom_{\mathsf{C}^{\mathrm{op}}}(X, Y) := \hom_{\mathsf{C}}(Y, X)$. For example, there are forgetful functors

$$U : \mathsf{Top} \longrightarrow \mathsf{Set}, \quad U : \mathsf{Grp} \longrightarrow \mathsf{Set}, \quad U : \mathsf{Ab} \longrightarrow \mathsf{Set}$$

which forget to the underlying category.

A natural transformation $\varphi : F \longrightarrow G$ between functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ is a map of functors. It consists of a morphism $\varphi_X : F(X) \longrightarrow G(X)$ for each object X of C such that the diagram

$$\begin{array}{ccc} X & \qquad & F(X) \xrightarrow{\varphi_X} G(X) \\ \Big\downarrow f & & F(f)\Big\downarrow \qquad\qquad \Big\downarrow G(f) \\ Y & & F(Y) \xrightarrow{\varphi_Y} G(Y) \end{array}$$

commutes for each morphism f of C. The intuition is that the maps φ_X are defined in a naturally occurring way. For example, let Y be a set and consider the functors defined objectwise by

$$- \times Y : \mathsf{Set} \longrightarrow \mathsf{Set}, \qquad X \longmapsto X \times Y,$$
$$\hom_{\mathsf{Set}}(Y, -) : \mathsf{Set} \longrightarrow \mathsf{Set}, \qquad X \longmapsto \hom_{\mathsf{Set}}(Y, X).$$

There is an evaluation map $e_X : \hom_{\mathsf{Set}}(Y, X) \times Y \longrightarrow X$ which is natural in X. In other words, the diagram

$$\begin{array}{ccc} X & \qquad & \hom_{\mathsf{Set}}(Y, X) \times Y \xrightarrow{e_X} X \\ \Big\downarrow f & & \Big\downarrow f_* \times \mathrm{id} \qquad\qquad \Big\downarrow f \\ Z & & \hom_{\mathsf{Set}}(Y, Z) \times Y \xrightarrow{e_Z} Z \end{array}$$

commutes for each morphism f of Set.

Define the *homotopy category* hTop with objects the topological spaces and with morphisms the homotopy classes of maps. In particular, $\hom_{\mathsf{hTop}}(X, Y) := [X, Y]$. Note that, by an exercise above, the composition pairing $[g] \circ [f] := [g \circ f]$ is well-defined. There is a functor p defined by

$$p : \mathsf{Top} \longrightarrow \mathsf{hTop}, \qquad X \xrightarrow{f} Y \quad \longmapsto \quad X \xrightarrow{[f]} Y$$

which is the identity on objects and which sends each map f to its homotopy class $[f]$. The homotopy category hTop is a quotient category in the sense of the following universal property.

Exercise 10.

(a) Prove that p identifies homotopic maps. In other words, prove that $p(f) = p(g)$ whenever $f \simeq g$.

(b) Prove that p satisfies the following universal property: given any functor $F : \mathsf{Top} \longrightarrow \mathsf{C}$ such that $F(f) = F(g)$ whenever $f \simeq g$, there exists a unique functor $\overline{F}$ which makes the diagram

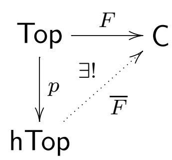
$$\begin{array}{ccc} \mathsf{Top} & \xrightarrow{F} & \mathsf{C} \\ \Big\downarrow p & \exists!\ \nearrow \overline{F} & \\ \mathsf{hTop} & & \end{array}$$

commute.

The homotopy category hTop satisfies another universal property. The following exercise shows that hTop is the "localization" of Top with respect to the collection of all homotopy equivalences.

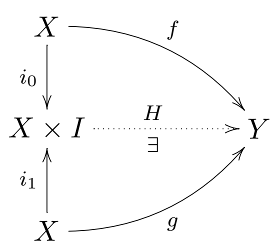
Exercise 11.

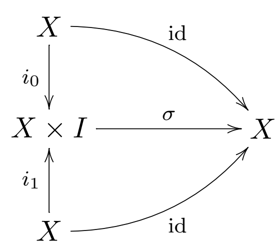
(a) Prove that p sends homotopy equivalences to isomorphisms.
(b) Prove that p satisfies the following universal property: given any functor $F : \mathsf{Top} \longrightarrow \mathsf{C}$ which sends homotopy equivalences to isomorphisms, there exists a unique functor $\overline{F}$ which makes the diagram



commute.

Checking part (b) amounts to verifying that: if a functor $F : \mathsf{Top} \longrightarrow \mathsf{C}$ sends homotopy equivalences to isomorphisms, then F identifies homotopic maps. Here is one approach: consider any $f, g : X \longrightarrow Y$ in Top such that $f \simeq g$. We want to show that $F(f) = F(g)$. We know the following diagrams





commute. Note that the projection map σ is a homotopy equivalence, apply the functor F to both of the diagrams, and argue that $F(f) = F(g)$.

Let Y, Z be spaces and define the mapping space $\mathrm{Map}(Y, Z)$ to be the set $\hom_{\mathsf{Top}}(Y, Z)$ equipped with the compact-open topology. Sometimes the space of maps $\mathrm{Map}(Y, Z)$ is denoted by the exponential notation Z^Y. Recall the following properties of this mapping space; sometimes referred to as the exponential law.

Proposition 12. *Let X, Y, Z be spaces. If Y is Hausdorff and locally compact, then there are isomorphisms of sets*

$$\hom_{\mathsf{Top}}(X \times Y, Z) \cong \hom_{\mathsf{Top}}(X, \mathrm{Map}(Y, Z))$$

natural in such X, Y, Z.

Let X be a space and consider any $x, y \in X$. Define $x \sim y$ if there exists a map $f : I \longrightarrow X$ such that $f(0) = x$ and $f(1) = y$. In other words, $x \sim y$ if and only if there is a path in X which starts at x and ends at y.

Exercise 13. Prove that $\sim$ defines an equivalence relation on X.

Hence X is partitioned into equivalence classes via $\sim$. The equivalence classes are called *path components* of X, and the set of all path components is denoted by $\pi_0(X)$. If $x \in X$, then the path component of x is denoted by $[x]$. If $f : X \longrightarrow Y$ is a map of spaces, define $\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$ to be the function sending a path component C of X to the path component of Y containing $f(C)$.

Exercise 14.

(a) Prove that $\pi_0 : \mathsf{Top} \longrightarrow \mathsf{Set}$ is a well-defined functor.

(b) Prove that π_0 factors uniquely through the homotopy category hTop. In other words, prove that π_0 fits into a commutative diagram of the form

$$\begin{array}{ccc} \mathsf{Top} & \xrightarrow{\pi_0} & \mathsf{Set} \\ \Big\downarrow p & \overset{\exists!}{\nearrow}{}_{\pi_0} & \\ \mathsf{hTop} & & \end{array}$$

(c) Use part (b) to prove that if two spaces X and Y have the same homotopy type, then $\pi_0(X) \cong \pi_0(Y)$.

Exercise 15.

(a) Let X be a space. Prove that $\pi_0(X) \cong [*, X]$.

(b) Let X be a based space. Prove that $\pi_0(X) \cong [S^0, X]_*$.

(c) Let X be a space which is Hausdorff and locally compact. Prove that $\pi_0(\mathrm{Map}(X, Y)) \cong [X, Y]$.

Part (b) is where the functor π_0 gets its name.

Here are some references for this material: [4, Chapter 1], [2, Chapters 1-2], [1, Chapter 1-2], [3, Chapter 0-1].

References

[1] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic topology from a homotopical viewpoint*. Universitext. Springer-Verlag, New York, 2002. Translated from the Spanish by Stephen Bruce Sontz.

[2] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. Available at: `http://www.math.uchicago.edu/ may/` .

[3] Joseph J. Rotman. *An introduction to algebraic topology*, volume 119 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.

[4] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York, 1981. Corrected reprint.