Elements of Homotopy **Fall 2008**

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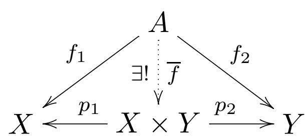
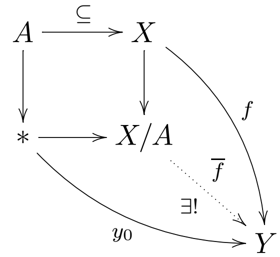
Series 3

Recall that if X, Y are spaces, then the product $X \times Y$ satisfies the universal property: given two continuous maps $f_1 : A \longrightarrow X$ and $f_2 : A \longrightarrow Y$, then there exists a unique continuous map $\overline{f}$ such that the diagram



commutes. Here p_1, p_2 are the obvious projection maps. By the universal property, it is easy to map a space A into $X \times Y$. Sometimes the map $\overline{f}$ is denoted by $(f_1, f_2) : A \longrightarrow X \times Y$.

Recall that if X is a space and $A \subseteq X$ is a non-empty subspace, then the quotient space X/A (obtained from X by collapsing A to a point) satisfies the universal property: given a continuous map $f : X \longrightarrow Y$ and a point $y_0 \in Y$ such that the outer diagram commutes, then there exists a unique continuous map $\overline{f}$ such that the diagram



commutes. Note, the outer diagram commutes if and only if $f(A) = \{y_0\}$. By the universal property, it is easy to map out of X/A into a space Y.

Let X be a based space. Since $I^n/\partial I^n \cong S^n$ for $n \geq 1$, it follows (from the case $n = 1$) that there are isomorphisms

(1) $$\pi_1(X) \cong [I/\partial I, X]_* \cong [S^1, X]_*.$$

natural in X.

Exercise 1. Let X, Y be based spaces.

(a) Prove that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Exercise 2. Let X be a discrete space. Prove that $\pi_1(X) = 0$.

Let $A \subseteq X$ be a subspace and let $i : A \longrightarrow X$ be the inclusion map. Then A is a *strong deformation retract* of X if there is a continuous map $r : X \longrightarrow A$ such that $r \circ i = \text{id}$ and there exists a homotopy $H : X \times I \longrightarrow X$ from $i \circ r$ to id such that $H(a, t) = a$ for each $a \in A$ and $t \in I$. Note, if A is a strong deformation retract of a non-empty space X, then, in particular, $A \simeq_* X$ and $A \simeq X$.

As we will see next week, π_1 is a homotopy invariant, i.e., if $(X, x_0) \simeq_* (Y, y_0)$ then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

Exercise 3. Let $A \subseteq \mathbb{R}^n$ be a convex subspace and consider any point $a_0 \in A$. Use a straight-line homotopy to prove that $\{a_0\}$ is a strong deformation retract of A.

Exercise 4. Use Exercise 3 to prove the following:

(a) $\pi_1(\mathbb{R}^n) \cong \pi_1(*) = 0$.
(b) $\pi_1(D^n) \cong \pi_1(*) = 0$.

Here $*$ denotes a 1-point set.

Exercise 5.

(a) Prove that S^n is a strong deformation retract of $\mathbb{R}^{n+1} - \{0\}$.
(b) Prove that $\pi_1(S^n) \cong \pi_1(\mathbb{R}^{n+1} - \{0\})$.
(c) Prove that $\pi_0(S^k) \cong \pi_0(\mathbb{R}^{k+1} - \{0\}) = *$ for $k \geq 1$. In particular, S^k is path connected for each $k \geq 1$.

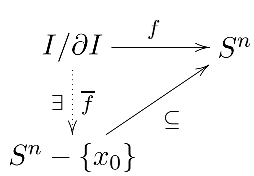
Exercise 6. Consider $S^n \subseteq \mathbb{R}^{n+1}$ and let $p := (1, 0, \ldots, 0) \in S^n$.

(a) Use stereographic projection to prove that $S^n - \{p\} \cong \mathbb{R}^n$.
(b) Prove that $\pi_1(S^n - \{p\}) \cong \pi_1(\mathbb{R}^n) = 0$.
(c) Prove that $\pi_0(S^n - \{p\}) = *$ and hence $S^n - \{p\}$ is path connected.

The space $S^n - \{p\}$ is sometimes called a *punctured* $n - $*sphere*.

Exercise 7. Let $[f] \in \pi_1(S^n)$. Prove the following:

(a) If $f : I/\partial I \longrightarrow S^n$ is a non-surjective map, then $[f] = [0] \in \pi_1(S^n)$. To prove this, note that f fits into a commutative diagram of the form



for some $x_0 \in S^n$ and use Exercise 6.
(b) It is a fact (which we will see later) that for $n \geq 2$, any continuous map $f : I/\partial I \longrightarrow S^n$ is based homotopic to a non-surjective continuous map. Assuming this fact, use (a) to prove that $\pi_1(S^n) = 0$ for $n \geq 2$.

Remark: The Seifert-van Kampen theorem (which we will see later) can be used to give a different proof that $\pi_1(S^n) = 0$ for $n \geq 2$.

Exercise 8. Let Y be a based space and consider any based continuous map $f : S^1 \longrightarrow Y$. Prove that $[f] = [0] \in \pi_1(Y)$ if and only if there exists a continuous map $\overline{f}$ which makes the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \exists \nearrow {\scriptstyle \overline{f}} & \\ D^2 & & \end{array}$$

commute. See [Exercise 8, Series 1 & 2].

Exercise 9. Let Y be a space. Prove that the following are equivalent:

(a) Every continuous map $f : S^1 \longrightarrow Y$ is homotopic to a constant map

(b) For each continuous map $f : S^1 \longrightarrow Y$, there exists a continuous map $\overline{f}$ which makes the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \exists \nearrow {\scriptstyle \overline{f}} & \\ D^2 & & \end{array}$$

commute

(c) $\pi_1(Y, y_0) = 0$ for every $y_0 \in Y$.

See [Exercise 8, Series 1 & 2].

A space Y is called *simply connected* if it is path connected and $\pi_1(Y, y_0) = 0$ for each $y_0 \in Y$.

Exercise 10. Prove that a space Y is simply connected if and only if all continuous maps $S^1 \longrightarrow Y$ are homotopic.

Exercise 11. Prove that every contractible space Y is simply connected. See [Exercise 9, Series 1 & 2].

Here are some references for this material: [1, Chapter 1], [2, Chapter 1], [3, Chapters 1].

References

[1] Brayton Gray. *Homotopy theory.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. An introduction to algebraic topology, Pure and Applied Mathematics, Vol. 64.

[2] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[3] Edwin H. Spanier. *Algebraic topology.* Springer-Verlag, New York, 1981. Corrected reprint.