Elements of Homotopy **Fall 2008**

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Series 4

Exercise 1. Let X be a based space, $x_0, x_1 \in X$, and consider any continuous map $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$; i.e., α is a path in X from x_0 to x_1. The purpose of this exercise is to prove the naturality of base point change. In other words, given any continuous map $f : X \longrightarrow Y$, prove that the diagram

$$\begin{array}{ccccc} X & \qquad & \pi_1(X, x_0) & \xrightarrow{\alpha^{\#}} & \pi_1(X, x_1) \\ {\scriptstyle f}\downarrow & & {\scriptstyle \pi_1(f)}\downarrow & & \downarrow{\scriptstyle \pi_1(f)} \\ Y & & \pi_1(Y, f(x_0)) & \xrightarrow{(f\alpha)^{\#}} & \pi_1(Y, f(x_1)) \end{array}$$

commutes.

A *topological group* is a based space X together with three based maps

$$\begin{array}{rl} m : X \times X \longrightarrow X & \text{“multiplication”} \\ e : * \longrightarrow X & \text{“two-sided unit”} \\ \nu : X \longrightarrow X & \text{“inverse”} \end{array}$$

which make the following diagrams

(1)
$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times \mathrm{id}} & X \times X \\ \downarrow{\scriptstyle \mathrm{id} \times m} & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow{m} & X \end{array} \qquad \begin{array}{ccccc} * \times X & \xrightarrow{e \times \mathrm{id}} & X \times X & \xleftarrow{\mathrm{id} \times e} & X \times * \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \cong} \\ X & = & X & = & X \end{array}$$

(2)
$$\begin{array}{ccccc} X & \xrightarrow{(\nu, \mathrm{id})} & X \times X & \xleftarrow{(\mathrm{id}, \nu)} & X \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ * & \xrightarrow{e} & X & \xleftarrow{e} & * \end{array}$$

in Top_* commute.

Exercise 2. Let X be a topological group. For any $x, y \in X$ define $x \cdot y := m(x, y)$ and $x^{-1} := \nu(x)$.

(a) Verify that the left-hand diagram in (1) commutes if and only if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for every $x, y, z \in X$.

(b) Verify that the right-hand diagram in (1) commutes if and only if $e \cdot x = x = x \cdot e$ for every $x \in X$.

(c) Verify that the diagram in (2) commutes if and only if $x^{-1} \cdot x = e = x \cdot x^{-1}$ for every $x \in X$.

In particular, a group X is exactly a topological group X whose underlying topology is discrete (i.e., as a space, X is a set).

Exercise 3. Let G be a topological group. Consider the composition of group homomorphisms

$$\pi_1(G,e) \times \pi_1(G,e) \xrightarrow{\cong} \pi_1(G \times G, (e,e)) \xrightarrow{\pi_1(m)} \pi_1(G,e)$$

$$\pi_1(G,e) \times \pi_1(G,e) \xrightarrow{\pi_1(m)} \pi_1(G,e)$$

which we also denote by $\pi_1(m)$. Consider any $[f]_*, [g]_* \in \pi_1(G,e)$.

(a) Prove that

$$\pi_1(m)([f]_*, [e]_*) = [f]_*$$
$$\pi_1(m)([e]_*, [g]_*) = [g]_*$$

(b) Prove that

$$([f]_*, [g]_*) = ([f]_*, [e]_*) \star ([e]_*, [g]_*)$$
$$([f]_*, [g]_*) = ([e]_*, [g]_*) \star ([f]_*, [e]_*)$$

(c) Use (a) and (b) to prove that $[f]_* \star [g]_* = [g]_* \star [f]_*$.
(d) Conclude that if G is a topological group, then $\pi_1(G,e)$ is abelian.
(e) Prove that $\pi_1(S^1)$ is abelian.
(f) Prove that $\pi_1(T)$ of the torus $T = S^1 \times S^1$ is abelian.

Exercise 4. Let A be a based space.

(a) Prove that $[A, *]_* \cong *$.
(b) Prove that $[A, X \times Y]_* \cong [A, X]_* \times [A, Y]_*$ for any based spaces X, Y.
(c) Let X be a topological group. Prove that the topological group structure on X induces a group structure on the set $[A, X]_*$.

Note: Part (c) can be proved by applying the functor $[A, -]_* : \mathsf{Top}_* \longrightarrow \mathsf{Set}_*$ to the commutative diagrams (1) and (2) in Top_* and using (a) and (b). Here, Set_* is the category of based sets and basepoint preserving functions.

The conclusion of Exercise 4 remains true with weaker conditions on the space X. An *H-group* is a based space X together with three based maps

$$m : X \times X \longrightarrow X \qquad \text{"multiplication"}$$
$$e : * \longrightarrow X \qquad \text{"two-sided unit"}$$
$$\nu : X \longrightarrow X \qquad \text{"inverse"}$$

which make the diagrams in (1) and (2) commute up to based homotopy; i.e., such that

$$m \circ (m \times \mathrm{id}) \simeq_* m \circ (\mathrm{id} \times m) \tag{3}$$

$$m \circ (e \times \mathrm{id}) \simeq_* \mathrm{id} \simeq_* m \circ (\mathrm{id} \times e) \tag{4}$$

$$m \circ (\nu, \mathrm{id}) \simeq_* e \simeq_* m \circ (\mathrm{id}, \nu). \tag{5}$$

Define the quotient category $\mathsf{hTop}_* := \mathsf{Top}_* / \simeq_*$, similar to the way we defined the quotient category $\mathsf{hTop} := \mathsf{Top} / \simeq$. The conditions (3)-(5) are the same as requiring that the diagrams in (1) and (2) commute in hTop_*. In particular, every topological group is an H-group.

Exercise 5. Let A be a based space.

(a) Let X be an H-group. Prove that the H-group structure on X induces a group structure on the set $[A, X]_*$.

Note: Part (a) can be proved by applying the functor $[A, -]_* : \mathsf{hTop}_* \longrightarrow \mathsf{Set}_*$ to the commutative diagrams (1) and (2) in hTop_* and using (a) and (b) from Exercise 4.

The conclusion of Exercise 3 remains true under weaker conditions. An *H-space* is a based space X together with two based maps

$$\begin{array}{rll} m : X \times X \longrightarrow X & \qquad & \text{“multiplication”} \\ e : * \longrightarrow X & \qquad & \text{“two-sided unit”} \end{array}$$

which make the diagrams

$$\begin{array}{ccccc} * \times X & \xrightarrow{e \times \mathrm{id}} & X \times X & \xleftarrow{\mathrm{id} \times e} & X \times * \\ \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle \cong} \\ X & = & X & = & X \end{array}$$

commute up to based homotopy; i.e., such that the diagrams commute in hTop_*. In particular, every H-group is an H-space.

Exercise 6. Let X be an H-space. Prove that $\pi_1(X, e)$ is abelian.

Given a set S, we would like to build a group $F(S)$ together with a function $i : S \longrightarrow F(S)$ which satisfies the universal property: given any group G and function $f : S \longrightarrow G$, then there exists a unique homomorphism of groups $\overline{f}$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & G \\ \Big\downarrow{\scriptstyle i} & \overset{\exists!}{\nearrow}{\scriptstyle \overline{f}} & \\ F(S) & & \end{array} \tag{6}$$

commutes. Such a group $F(S)$ is called the *free group* generated by S. The purpose of the next exercise is to prove that such a group $F(S)$ exists. The next exercise closely follows the argument in Chapter 6 of Artin's Algebra book. See also Grillet's Algebra book.

Exercise 7 (Existence of the free group $F(S)$)**.** Start with an arbitrary set S, say $S = \{a, b, c, \cdots\}$. We call the elements in S the *symbols* in S, and define a *word* to be a finite string of symbols from S. Examples of words include a, ba, cca, and $bacb$. Two words can be concatenated:

$$ba, cca \longmapsto bacca$$

If we allow the "empty word", then concatenation of words has a two-sided unit. Let's denote the empty word by the symbol 1. Hence if X denotes the set of all words, including the empty word, then concatenation of words together with the empty word 1 define two functions

$$\begin{aligned} m : X \times X &\longrightarrow X \qquad \text{"multiplication"} \\ 1 : * &\longrightarrow X \qquad \text{"two-sided unit"} \end{aligned}$$

such that the diagrams in (1) commute. More formally, the set of words is defined by $X = \amalg_{t \geq 0} S^{\times t}$ with $S^{\times 0} = * = \{1\}$. Here, $\amalg$ denotes the disjoint union of sets. The set X is called the *free monoid* generated by the set S. Unfortunately, X is not a group because it does not have inverses.

To construct the free group $F(X)$, the idea is to *formally add inverses* to the set of words X. Let S' be the set consisting of the symbols in S and also of symbols a^{-1} for every $a \in S$:

$$S' = \{a, a^{-1}, b, b^{-1}, c, c^{-1}, \dots\}.$$

Define X' to be the set of words made using the symbols in S', including the empty word. More formally, $X' = \amalg_{t \geq 0} (S \amalg S)^t$ with $(S \amalg S)^{\times 0} = * = \{1\}$. If a word $x \in X'$ looks like

$$\cdots zz^{-1} \cdots \quad \text{or} \quad \cdots z^{-1}z \cdots$$

for some $z \in S$, the we can agree to *cancel* the two symbols z and z^{-1} and reduce the length of the word. A word $x \in X'$ will be called *reduced* if no such cancellation can be made. Starting with any word x, we can perform a finite sequence of cancellations and eventually get a reduced word x_0, possibly the empty word 1. We call this word x_0 a *reduced form* of x.

(a) Consider the word $x = babb^{-1}a^{-1}c^{-1}ca$. Show that there are at least two different sequences of cancellations which give the reduced word ba.

(b) Prove that there is only one reduced form of a given word x.

Two words $x, x' \in X'$ are *equivalent*, denoted $x \sim x'$, if they have the same reduced form. This defines an equivalence relation $\sim$ on the set X'.

(c) Prove that the product of equivalent words is equivalent: if $x \sim x'$ and $y \sim y'$, then $xy \sim x'y'$.

Define $F(S) := X'/\sim$. Then an element of $F(S)$ corresponds to exactly one reduced word in X'. There are three maps

$$\begin{aligned} m : F(S) \times F(S) &\longrightarrow F(S) \qquad \text{"multiplication"} \\ e : * &\longrightarrow F(S) \qquad \text{"two-sided unit"} \\ \nu : F(S) &\longrightarrow F(S) \qquad \text{"inverse"} \end{aligned}$$

which give $F(S)$ the structure of a group. To multiply reduced words, concatenate them and cancel:

$$(abc^{-1})(ca) \longmapsto abc^{-1}ca = aba.$$

Define the function $i : S \longrightarrow F(S)$ by $i(s) := s$ the natural inclusion.

(d) Prove that $F(S)$ satisfies the universal property (6).

(e) Prove that $F(\emptyset) = 0$ and $F(*) \cong \mathbb{Z}$.