Elements of Homotopy **Fall 2008**

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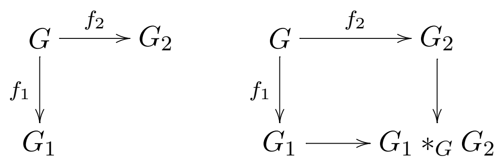
Series 5

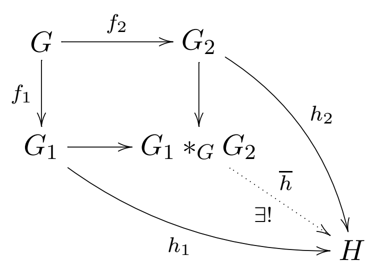
Let G_1, G_2, G be groups and suppose $f_1 : G \longrightarrow G_1$ and $f_2 : G \longrightarrow G_2$ are homomorphisms of groups. We can organize this data into the left-hand diagram

(1)



in the category Grp of groups. The *amalgamated product* of G_1 and G_2 over G, written $G_1 *_G G_2$, is a group, together with the right-hand commutative diagram in (1), which satisfies the universal property: given two homomorphisms h_1, h_2 such that the outer diagram in (2) commutes, then there exists a unique homomorphism $\overline{h}$ such that the diagram

(2)



commutes.

The purpose of the following exercise is to prove existence of amalgamated products.

Exercise 1. Define the group $G_1 *_G G_2 := (G_1 * G_2)/\sim$, the free product of groups G_1 and G_2 modulo $f_1(g) \sim f_2(g)$, $g \in G$. More precisely, define $G_1 *_G G_2 := (G_1 * G_2)/N$ where N is the normal subgroup generated by $f_1(g)(f_2(g))^{-1}$ for all $g \in G$.

(a) Prove that $G_1 *_G G_2$ satisfies the universal property of amalgamated products.

Exercise 2.

(a) Use the universal property to prove that $G_1 *_G G_2 \cong G_2 *_G G_1$.

(b) Denote by e the trivial group. Use the universal property to prove that $G_1 * G_2 \cong G_1 *_e G_2$.

Exercise 3. Let G and H be groups and consider homomorphisms of the form

$$\begin{array}{ccc} G & \xrightarrow{f_2} & H \\ \cong \big\downarrow f_1 & & \\ G & & \end{array}$$

such that f_1 is an isomorphism.

(a) Use the universal property to prove that $G *_G H \cong H$.

Exercise 4. Let $n \geq 1$. Consider the set of symbols $\{a_1, a_2, \cdots, a_n\}$ which has exactly n-elements.

(a) Prove that $\mathbb{Z} * \mathbb{Z} \cong F(a_1, a_2)$ the free group generated by the set $\{a_1, a_2\}$.
(b) Prove that the free product (n-copies) $\mathbb{Z} * \cdots * \mathbb{Z} \cong F(a_1, \cdots, a_n)$ the free group generated by the set $\{a_1, \cdots, a_n\}$.

Let X be a space and suppose $X_1, X_2 \subseteq X$ are open subspaces such that $X = X_1 \cup X_2$ and there exists a point $* \in X_1 \cap X_2$. We can organize this data into a commutative diagram of inclusions of the form

(3)
$$\begin{array}{ccc} X_1 \cap X_2 & \longrightarrow & X_2 \\ \big\downarrow & & \big\downarrow \\ X_1 & \longrightarrow & X \end{array}$$

in Top_*. Applying the functor $\pi_1 : \mathsf{Top}_* \longrightarrow \mathsf{Grp}$ to (3) gives a commutative diagram of the form

(4)
$$\begin{array}{ccc} \pi_1(X_1 \cap X_2) & \longrightarrow & \pi_1(X_2) \\ \big\downarrow & & \big\downarrow \\ \pi_1(X_1) & \longrightarrow & \pi_1(X) \end{array}$$

in Grp. The following is one form of the Seifert-van Kampen theorem that appears often.

Theorem 5 (Seifert-van Kampen theorem)**.** *Let X be a space and suppose $X_1, X_2 \subseteq X$ are open subspaces such that $X = X_1 \cup X_2$ and there exists a point $* \in X_1 \cap X_2$. If the subspaces $X_1, X_2, X_1 \cap X_2$ are path-connected, then*

$$\pi_1(X) \cong \pi_1(X_1) *_{\pi_1(X_1 \cap X_2)} \pi_1(X_2).$$

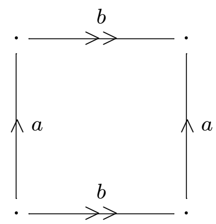
We will prove the following calculation later in the course.

Theorem 6. *$\pi_1(S^1) \cong \mathbb{Z}$.*

Exercise 7. Prove the following.

(a) $\pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.
(b) $\pi_1\big((S^1 \vee S^2) \times D^3 \times (S^1 \vee S^1 \vee D^2)\big) \cong Z \times (\mathbb{Z} * \mathbb{Z})$.

Exercise 8. Recall that the torus $T = S^1 \times S^1$ can be obtained from I^2 by glueing edges as indicated in the picture below. In other words, $T \cong I^2/\sim$.



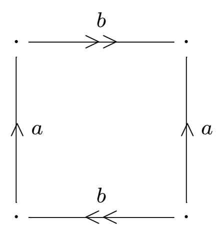
(a) Use the Seifert-van Kampen theorem together with $T \cong I^2/\sim$ to prove that

$$\pi_1(T) \cong F\{a, b\}/N$$

with N the normal subgroup generated by the word $aba^{-1}b^{-1}$.

(b) Prove that $F\{a, b\}/N \cong \mathbb{Z} \times \mathbb{Z}$.

Exercise 9. Recall that the Klein bottle K can be obtained from I^2 by glueing edges as indicated in the picture below. In other words, $K \cong I^2/\sim$.



(a) Use the Seifert-van Kampen theorem together with $K \cong I^2/\sim$ to prove that

$$\pi_1(K) \cong F\{a, b\}/N$$

with N the normal subgroup generated by the word $aba^{-1}b$.

Here are some references for this material: [1, Chapter 7].

References

[1] Brayton Gray. *Homotopy theory.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. An introduction to algebraic topology, Pure and Applied Mathematics, Vol. 64.