Elements of Homotopy **Fall 2008**

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If $p : E \longrightarrow B$ is a covering map, then E is called a *covering space* of B.

Definition 1. Let G be a group, X a space, and define $\mathrm{Homeo}(X, X)$ to be the set of all homeomorphisms from X to X.

- An *action* of G on X is a homomorphism of groups $G \longrightarrow \mathrm{Homeo}(X, X)$. It is common to use the notation $g \longmapsto (g : X \longrightarrow X, x \longmapsto gx)$.
- A *G-space* is a space X together with an action of G.
- If X is a G-space, the *orbit space* X/G is the quotient space $X/\sim$ such that $\sim$ is generated by the relations $gx \sim x$, for each $g \in G$ and $x \in X$. In particular, the natural projection map $p : X \longrightarrow X/G$ is continuous.
- An action of G on X is a *covering space action* (or properly discontinuous action) if each $x \in X$ has a neighborhood U such that the following condition is satisfied:

$$gU \cap U \neq \emptyset \quad \Longrightarrow \quad g = e. \tag{1}$$

  Here, $e \in G$ denotes the identity element and $g \in G$.

Covering space actions are useful for building covering maps.

Theorem 2. *Let G be a group and X a G-space. Assume the action of G on X is a covering space action.*

(a) *The projection $p : X \longrightarrow X/G$ is a covering map.*
(b) *If X is simply connected, then $\pi_1(X/G) \cong G$.*

The purpose of the following exercise is to prove Theorem 2(a); part (b) will be proved later.

Exercise 3. Assume the conditions of Theorem 2.

(a) Let $U \subseteq X$ be a subset. Prove that

$$p^{-1}\big(p(U)\big) = \bigcup_{g \in G} gU.$$

(b) Let $U \subseteq X$ be open. Prove that $p^{-1}\big(p(U)\big) \subseteq X$ is open.
(c) Prove that p is an open map.
(d) Prove that condition (1) is equivalent to the following:

$$g_1U \cap g_2U \neq \emptyset \quad \Longrightarrow \quad g_1 = g_2.$$

In particular, all the images gU for varying $g \in G$ are disjoint.

(d) Prove that p is a covering map.

The purpose of this exercise is to use Theorem 2 to construct several examples of covering maps.

Exercise 4.

(a) Find an action of $\mathbb{Z}$ on $\mathbb{R}$ such that the orbit space $\mathbb{R}/\mathbb{Z} \cong S^1$. Prove that $p : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ is a covering map. Hence, $\mathbb{R}$ is a simply connected covering space of the circle S^1.

(b) Find an action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ such that the orbit space $\mathbb{R}^2/\mathbb{Z}^2 \cong T$. Prove that $p : \mathbb{R}^2 \longrightarrow \mathbb{Z}^2$ is a covering map. Hence, $\mathbb{R}^2$ is a simply connected covering space of the torus $T = S^1 \times S^1$.

(c) Let $n \geq 2$. The antipodal map $S^n \longrightarrow S^n$, $x \longmapsto -x$, defines an action of $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ on the n-sphere S^n with orbit space $S^n/\mathbb{Z}_2 \cong \mathbb{R}P^n$, real projective n-space. Prove that $p : S^n \longrightarrow S^n/\mathbb{Z}_2$ is a covering map. Hence, S^n is a simply connected covering space of $\mathbb{R}P^n$.

(d) Construct a simply connected covering space of $S^1 \vee S^2$. Hint: consider an "infinite string of ballons" equipped with an action of $\mathbb{Z}$ similar to the action of $\mathbb{Z}$ on $\mathbb{R}$ in part (a).

(e) Construct a covering space of $S^1 \vee S^1$ as in part (d) by using 1-spheres instead of 2-spheres. In other words, consider an "infinite string of circles" equipped with an action of $\mathbb{Z}$.

Pictures of several more covering spaces for $S^1 \vee S^1$ appear in [1, 1.3].

Definition 5. A continuous map $p : E \longrightarrow B$ is a *local homeomorphism* if each point $e \in E$ has a neighborhood that is mapped homeomorphically by p onto an open subset of B.

Exercise 6. Let $p : E \longrightarrow B$ be a covering map.

(a) Prove that the fiber $p^{-1}(b) \subseteq E$ is a discrete subspace of E, for every point $b \in B$.

(b) Prove that every local homeomorphism is an open map.

(c) Prove that p is a local homeomorphism, and hence an open map.

Exercise 7.

(a) Find a continuous surjective map that is not a covering map.

(b) Find a local homeomorphism that is not a covering map.

For help with Exercise 7(b), see [3, Example 9.53.2].

Exercise 8. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ be covering maps.

(a) Prove that every homeomorphism is a covering map.

(b) Prove that $p \times p' : E \times E' \longrightarrow B \times B'$ is a covering map.

(c) Prove that $p \amalg p' : E \amalg E' \longrightarrow B \amalg B'$ is a covering map.

(d) Prove that an arbitrary disjoint union of covering maps is a covering map.

Exercise 8(b) is not true for arbitrary products of covering maps; see, for example, [4, 2.2.9].

Exercise 9. Let $p : X \longrightarrow Y$ and $q : Y \longrightarrow Z$ be covering maps. Assume that the fiber $q^{-1}(z) \subseteq Y$ is finite for each $z \in Z$.

(a) Prove that the composition $X \longrightarrow Y \longrightarrow Z$ is a covering map.

Exercise 9(a) is not true when the finiteness condition on fibers is dropped; see, for example, [4, 2.2.8].

Here are some references for this material: [1, Section 1.3], [2, Chapter 17], [3, Section 53], [4, Chapter 2].

References

[1] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. Available at `http://www.math.cornell.edu/~hatcher/` .

[2] Czes Kosniowski. *A first course in algebraic topology.* Cambridge University Press, Great Britain, 1980.

[3] James R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[4] Edwin H. Spanier. *Algebraic topology.* Springer-Verlag, New York, 1981. Corrected reprint.