If \( p : E \longrightarrow B \) is a covering map, then \( E \) is called a \textit{covering space} of \( B \).

**Definition 1.** Let \( G \) be a group, \( X \) a space, and define \( \text{Homeo}(X, X) \) to be the set of all homeomorphisms from \( X \) to \( X \).

- An \textit{action} of \( G \) on \( X \) is a homomorphism of groups \( G \to \text{Homeo}(X, X) \).
  It is common to use the notation \( g \mapsto (g : X \longrightarrow X, x \mapsto gx) \).
- A \textit{G-space} is a space \( X \) together with an action of \( G \).
- If \( X \) is a \( G \)-space, the \textit{orbit space} \( X/G \) is the quotient space \( X/\sim \) such that \( \sim \) is generated by the relations \( gx \sim x \), for each \( g \in G \) and \( x \in X \). In particular, the natural projection map \( p : X \longrightarrow X/G \) is continuous.
- An action of \( G \) on \( X \) is a \textit{covering space action} (or properly discontinuous action) if each \( x \in X \) has a neighborhood \( U \) such that the following condition is satisfied:

\[
gU \cap U \neq \emptyset \quad \Longrightarrow \quad g = e.
\]

Here, \( e \in G \) denotes the identity element and \( g \in G \).

Covering space actions are useful for building covering maps.

**Theorem 2.** Let \( G \) be a group and \( X \) a \( G \)-space. Assume the action of \( G \) on \( X \) is a covering space action.

(a) The projection \( p : X \longrightarrow X/G \) is a covering map.

(b) If \( X \) is simply connected, then \( \pi_1(X/G) \cong G \).

The purpose of the following exercise is to prove Theorem 2(a); part (b) will be proved later.

**Exercise 3.** Assume the conditions of Theorem 2.

(a) Let \( U \subseteq X \) be a subset. Prove that

\[
p^{-1}(p(U)) = \bigcup_{g \in G} gU.
\]

(b) Let \( U \subseteq X \) be open. Prove that \( p^{-1}(p(U)) \subseteq X \) is open.

(c) Prove that \( p \) is an open map.

(d) Prove that condition (1) is equivalent to the following:

\[
g_1U \cap g_2U \neq \emptyset \quad \Longrightarrow \quad g_1 = g_2.
\]

In particular, all the images \( gU \) for varying \( g \in G \) are disjoint.

(d) Prove that \( p \) is a covering map.

The purpose of this exercise is to use Theorem 2 to construct several examples of covering maps.
Exercise 4.

(a) Find an action of \( \mathbb{Z} \) on \( \mathbb{R} \) such that the orbit space \( \mathbb{R}/\mathbb{Z} \cong S^1 \). Prove that \( p : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \) is a covering map. Hence, \( \mathbb{R} \) is a simply connected covering space of the circle \( S^1 \).

(b) Find an action of \( \mathbb{Z}^2 \) on \( \mathbb{R}^2 \) such that the orbit space \( \mathbb{R}^2/\mathbb{Z}^2 \cong T \). Prove that \( p : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 \) is a covering map. Hence, \( \mathbb{R}^2 \) is a simply connected covering space of the torus \( T = S^1 \times S^1 \).

(c) Let \( n \geq 2 \). The antipodal map \( S^n \longrightarrow S^n, \; x \mapsto -x \), defines an action of \( \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} \) on the \( n \)-sphere \( S^n \) with orbit space \( S^n/\mathbb{Z}_2 \cong \mathbb{RP}^n \), real projective \( n \)-space. Prove that \( p : S^n \longrightarrow S^n/\mathbb{Z}_2 \) is a covering map. Hence, \( S^n \) is a simply connected covering space of \( \mathbb{RP}^n \).

(d) Construct a simply connected covering space of \( S^1 \lor S^2 \). Hint: consider an “infinite string of ballons” equipped with an action of \( \mathbb{Z} \) similar to the action of \( \mathbb{Z} \) on \( \mathbb{R} \) in part (a).

(e) Construct a covering space of \( S^1 \lor S^1 \) as in part (d) by using 1-spheres instead of 2-spheres. In other words, consider an “infinite string of circles” equipped with an action of \( \mathbb{Z} \).

Pictures of several more covering spaces for \( S^1 \lor S^1 \) appear in [1, 1.3].

Definition 5. A continuous map \( p : E \longrightarrow B \) is a local homeomorphism if each point \( e \in E \) has a neighborhood that is mapped homeomorphically by \( p \) onto an open subset of \( B \).

Exercise 6. Let \( p : E \longrightarrow B \) be a covering map.

(a) Prove that the fiber \( p^{-1}(b) \subseteq E \) is a discrete subspace of \( E \), for every point \( b \in B \).

(b) Prove that every local homeomorphism is an open map.

(c) Prove that \( p \) is a local homeomorphism, and hence an open map.

Exercise 7.

(a) Find a continuous surjective map that is not a covering map.

(b) Find a local homeomorphism that is not a covering map.

For help with Exercise 7(b), see [3, Example 9.53.2].

Exercise 8. Let \( p : E \longrightarrow B \) and \( p' : E' \longrightarrow B' \) be covering maps.

(a) Prove that every homeomorphism is a covering map.

(b) Prove that \( p \times p' : E \times E' \longrightarrow B \times B' \) is a covering map.

(c) Prove that \( p \amalg p' : E \amalg E' \longrightarrow B \amalg B' \) is a covering map.

(d) Prove that an arbitrary disjoint union of covering maps is a covering map.

Exercise 8(b) is not true for arbitrary products of covering maps; see, for example, [4, 2.2.9].

Exercise 9. Let \( p : X \longrightarrow Y \) and \( q : Y \longrightarrow Z \) be covering maps. Assume that the fiber \( q^{-1}(z) \subseteq Y \) is finite for each \( z \in Z \).
(a) Prove that the composition $X \to Y \to Z$ is a covering map.

Exercise 9(a) is not true when the finiteness condition on fibers is dropped; see, for example, [4, 2.2.8].

Here are some references for this material: [1, Section 1.3], [2, Chapter 17], [3, Section 53], [4, Chapter 2].

REFERENCES


