**Elements of Homotopy** **Fall 2008**
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## Series 8

Let $G$ be a group and $X$ a space. We can regard $G$ as a discrete space. Recall that a *left action* of $G$ on $X$ (or left $G$-action on $X$) is a continuous map

$$G \times X \longrightarrow X, \quad (g, x) \longmapsto gx$$

such that $g_1(g_2x) = (g_1g_2)x$ and $ex = x$ for every $g_1, g_2 \in G$ and $x \in X$. Here, $e \in G$ denotes the identity element. Sometimes it is convenient for the group $G$ to act on the right. A *right action* of $G$ on $X$ (or right $G$-action on $X$) is a continuous map

$$X \times G \longrightarrow X, \quad (x, g) \longmapsto xg$$

such that $(xg_1)g_2 = x(g_1g_2)$ and $xe = x$ for every $g_1, g_2 \in G$ and $x \in X$.

**Exercise 1.** Let $G$ be a group and $X$ a space.

(a) Prove that every left $G$-action on $X$ determines a right $G$-action on $X$ given by $xg := g^{-1}x$.
(b) Similarly, prove that every right $G$-action on $X$ determines a left $G$-action on $X$ given by $gx := xg^{-1}$.

Recall the unique homotopy lifting property of covering maps. We denote by $*$ a 1-point space.

**Theorem 2.** *Let $p : E \longrightarrow B$ be a covering map and let $Y = *$ or $Y = [0,1]$. Given any solid commutative diagram of the form*

(1)
$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d, "\subseteq"'] & E \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "H"'] \arrow[ur, dotted, "\widehat{H}", "\exists!"'] & B \end{tikzcd}$$

*in* Top*, there exists a unique continuous map $\widehat{H}$ such that* (1) *commutes.*

Theorem 2 remains true for any space $Y$; the proof is very similar to the argument given in lecture for $Y = [0,1]$.

**Exercise 3.** Let $p : E \longrightarrow B$ be a covering map, $b_0 \in B$, and consider the fiber $F := p^{-1}(b_0) \subseteq E$. A right action of $\pi_1(B, b_0)$ on the fiber $F$ is defined by lifting loops as follows:

(2)
$$F \times \pi_1(B, b_0) \longrightarrow F, \quad (e_0, [\lambda]) \longmapsto e_0[\lambda] := \widehat{\lambda}_{e_0}(1).$$

Here, $\widehat{\lambda}_{e_0} : [0,1] \longrightarrow E$ is the unique lift of the loop $\lambda : [0,1] \longrightarrow B$ such that $\widehat{\lambda}_{e_0}(0) = e_0$; in other words, $\widehat{\lambda}_{e_0}$ is the unique map which makes the diagram

$$\begin{tikzcd}
\{0\} \arrow[r, "e_0"] \arrow[d, "\subseteq"'] & E \arrow[d, "p"] \\
{[0,1]} \arrow[r, "\lambda"'] \arrow[ur, dotted, "\exists!", "\widehat{\lambda}_{e_0}"'] & B
\end{tikzcd}$$

commute.

(a) Prove that (2) is a well-defined function.
(b) Prove that (2) defines a right $\pi_1(B, b_0)$-action on the fiber $F$.

Part (a) can be argued using Theorem 2 and the unique path lifting property (the special case of Theorem 2 obtained by taking $Y = *$) to prove: if $[\lambda] = [\lambda']$, then $\widehat{\lambda}_{e_0}(1) = \widehat{\lambda'}_{e_0}(1)$.

**Exercise 4.** Recall the simply connected covering space of $S^1 \vee S^2$ given by an "infinite string of balloons". Denote by $p : E \longrightarrow S^1 \vee S^2$ the corresponding covering map .

(a) Describe geometrically the action of $\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$ on the fiber $F := p^{-1}(b_0)$ for any $b_0 \in S^1 \vee S^2$.

**Definition 5.** Let $G$ be a group and $X$ a space with a right $G$-action. Let $x \in X$.

- The *isotropy subgroup of* $x$ (or stabilizer of $x$) is the subgroup $G_x \subseteq G$ defined by $G_x := \{g \in G \mid xg = x\}$.
- The *orbit of* $x$ is the subset defined by $\mathrm{orbit}(x) := \{xg \mid g \in G\} \subseteq X$.
- $G$ acts *transitively* on $X$ if $\mathrm{orbit}(x) = X$ for each $x \in X$; i.e., if there is exactly one orbit.

**Exercise 6.** Let $p : E \longrightarrow B$ be a covering map and $b_0 \in B$. By Exercise 3 there is a right action of $\pi_1(B, b_0)$ on the fiber $F := p^{-1}(b_0) \subseteq E$ given by lifting loops. Prove the following:

(a) The isotropy subgroup of $e_0 \in F$ is the image of
$$\pi_1(p) : \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0).$$
(b) $\mathrm{orbit}(e_0) \cong \pi_1(B, b_0)/\,\mathrm{image}(\pi_1(p))$ as $\pi_1(B, b_0)$-sets.
(c) If $E$ is path connected, then $\pi_1(B, b_0)$ acts transitively on $F$.
(d) If $E$ is path connected, then $F \cong \pi_1(B, b_0)/\,\mathrm{image}(\pi_1(p))$ as $\pi_1(B, b_0)$-sets.
(e) If $E$ is simply connected, then $F \cong \pi_1(B, b_0)$ as $\pi_1(B, b_0)$-sets.
(f) If $E$ is simply connected and $|F| = q$ for some prime $q$, then
$$\pi_1(B, b_0) \cong \mathbb{Z}/q\mathbb{Z}.$$

**Exercise 7.** Let $n \geq 2$. Recall the covering map $p : S^n \longrightarrow \mathbb{R}P^n$, let $b_0 \in \mathbb{R}P^n$, and consider the fiber $F := p^{-1}(b_0) \subseteq S^n$.

(a) Verify that $|F| = 2$.

(b) How many groups have exactly 2 elements?

(c) Use Exercise 6(e) to prove that $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Recall the following definitions.

**Definition 8.** Let $G$ be a group and $X$ a space.

- An action of $G$ on $X$ is a *covering space action* (or properly discontinuous action) if each $x \in X$ has a neighborhood $U$ such that the following condition is satisfied:

$$gU \cap U \neq \emptyset \quad \Longrightarrow \quad g = e.$$

Here, $e \in G$ denotes the identity element and $g \in G$.

- An action of $G$ on $X$ is *free* if for each $x \in X$ the following condition is satisfied:

$$gx = x \quad \Longrightarrow \quad g = e.$$

Here, $e \in G$ denotes the identity element and $g \in G$.

**Exercise 9.** Let $G$ be a finite group and $X$ a Hausdorff space with an action of $G$. Prove the following:

(a) If the action of $G$ on $X$ is free, then the $G$-action on $X$ is a covering space action.

**Exercise 10.** Consider the 3-sphere $S^3$ regarded as the subspace $S^3 := \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\} \subseteq \mathbb{C}^2$. Let $p$ be prime to $q$ and define the map $h : S^3 \longrightarrow S^3$ by

$$h(z_0, z_1) := (\exp(2\pi i/p)z_0, \exp(2\pi iq/p)z_1).$$

(a) Prove that $h$ is a homeomorphism and that $h^p = \mathrm{id}$.

(b) Define an action of $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ on $S^3$ by

$$\mathbb{Z}_p \times S^3 \longrightarrow S^3, \quad ([n], (z_0, z_1)) \longmapsto [n](z_0, z_1) := h^n(z_0, z_1)$$

and prove that this action is free.

(c) Prove that $p : S^3 \longrightarrow S^3/\mathbb{Z}_p$ is a covering map. The orbit space $S^3/\mathbb{Z}_p$ is called a *lens space* and is denoted by $L(p, q)$.

(d) Verify that $L(2, 1) \cong \mathbb{R}P^3$.

Here are some references for this material: [1, Chapter III], [2, Section 1.3], [3, Chapter 17,18,19], [4, Chapter 9], [5, Chapter 10].

## REFERENCES

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