Elements of Homotopy **Fall 2008**
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Series 9 & 10

Recall from lecture the following calculation.

Theorem 1. *$\pi_1(S^1) \cong \mathbb{Z}$.*

We will use this calculation in the next exercise to prove:

Theorem 2 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Exercise 3. The purpose of this exercise is to prove Theorem 2. Let $n \geq 1$ and consider any polynomial $p(z) \in \mathbb{C}[z]$ of the form

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

We want to show that p has a root in $\mathbb{C}$. Suppose not; then $p(b) \neq 0$ for every $b \in \mathbb{C}$. We want to find a contradiction. Define $F : I \times [0, \infty) \longrightarrow S^1 \subseteq \mathbb{C}$ by

$$F(s, r) := \frac{p\big(r \exp(2\pi i s)\big)}{|p\big(r \exp(2\pi i s)\big)|} \frac{|p(r)|}{p(r)} \tag{1}$$

(a) Verify that F is a well-defined continuous function.
(b) Verify that $F(0, r) = 1 = F(1, r)$ for every $r \geq 0$.
(c) Observe that $F : I \times [0, r] \longrightarrow S^1 \subseteq \mathbb{C}$ determines a homotopy of loops based at 1 for each $r > 0$.
(d) Verify that $F(-, 0) : I \longrightarrow S^1 \subseteq \mathbb{C}$ is the constant loop based at 1.
(e) Conclude that $[F(-, r)] = [F(-, 0)] = 0 \in \pi_1(S^1, 1) \cong \mathbb{Z}$, for every $r \geq 0$.

Choose and fix any $r > \max\{1, |a_{n-1}| + \cdots + |a_1| + |a_0|\}$.

(f) Verify that for all $z \in \mathbb{C}$ satisfying $|z| = r$, the following is true:
$$|z^n| > |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|$$

For each $t \in [0, 1]$, define the polynomial $p_t(z) \in \mathbb{C}[z]$ by

$$p_t(z) := z^n + t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$$

(g) Verify that $p_t(b) \neq 0$ for every $b \in \mathbb{C}$ satisfying $|b| = r$.
(h) Define $G : I \times [0, 1] \longrightarrow S^1 \subseteq \mathbb{C}$ by replacing p with p_t in (1) as follows:
$$G(s, t) := \frac{p_t\big(r \exp(2\pi i s)\big)}{|p_t\big(r \exp(2\pi i s)\big)|} \frac{|p_t(r)|}{p_t(r)}$$
(i) Conclude that $[G(-, 1)] = n \in \pi_1(S^1, 1) \cong \mathbb{Z}$.
(j) Use part (e) to show that $n = 0$. This contradicts $n \geq 1$. Therefore p must have a root in $\mathbb{C}$. This completes the proof of Theorem 2.

We will use Theorem 1 in the next exercise to prove:

Theorem 4 (Brouwer's fixed point theorem for the 2-disk)**.** *Every continuous map $f : D^2 \longrightarrow D^2$ has a fixed point; i.e., there exists a point $x \in D^2$ such that $f(x) = x$.*

Exercise 5. The purpose of this exercise is to prove Theorem 4. Let $f : D^2 \longrightarrow D^2$ be a continuous map. We want to show that f has a fixed point. Suppose not; then $f(x) - x \neq 0$ for every $x \in D^2$.

(a) Define a function $r : D^2 \longrightarrow S^1$ by setting $r(x)$ to be the point on S^1 obtained from the intersection of the line segment from $f(x)$ to x extended to meet S^1. Hint: Draw a picture.

(b) Prove that r is a continuous function.

(c) Verify that r makes the following diagram

(2)
$$\begin{array}{ccc} S^1 & \xrightarrow{\text{id}} & S^1 \\ \subseteq \big\downarrow & \nearrow_{r} & \\ D^2 & & \end{array}$$

commute.

(d) Apply the functor $\pi_1 : \mathsf{Top}_* \longrightarrow \mathsf{Grp}$ to diagram (2) to obtain a (left-hand side) commutative diagram of the form

(3)
$$\begin{array}{ccc} \pi_1(S^1) & \xrightarrow{\text{id}} & \pi_1(S^1) \\ \big\downarrow & \nearrow & \\ \pi_1(D^2) & & \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ \big\downarrow & \nearrow & \\ 0 & & \end{array}$$

in Grp.

(e) Use Theorem 1 to verify that the left-hand diagram in (3) has the form of the right-hand diagram in (3).

(f) Conclude that $n = 0$ for every $n \in \mathbb{Z}$. This gives a contradiction. Therefore f must have a fixed point. This completes the proof of Theorem 4.

A similar argument proves the following:

Theorem 6 (Brouwer's fixed point theorem for the n-disk)**.** *Let $n \geq 2$. Every continuous map $f : D^n \longrightarrow D^n$ has a fixed point; i.e., there exists a point $x \in D^n$ such that $f(x) = x$.*

Let X be a based space and recall that

$$\pi_0(X) \cong [S^0, X]_*$$
$$\pi_1(X) \cong [S^1, X]_* \cong [I/\partial I, X]_*$$

We will soon be studying the *higher homotopy groups* of X which satisfy

$$\pi_n(X) \cong [S^n, X]_* \cong [I^n/\partial I^n, X]_*,$$

for each $n \geq 2$. The following exercise is a useful application of the higher homotopy group functors π_n.

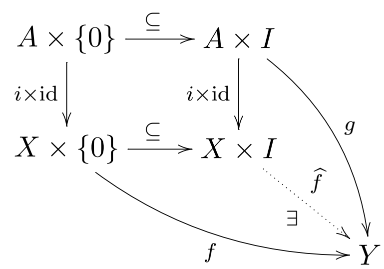
Exercise 7. Let $n \geq 1$. Use the fact that $\pi_n : \mathsf{Top}_* \longrightarrow \mathsf{Grp}$ is a well-defined functor and $\pi_n(S^n) \cong \mathbb{Z}$ for each $n \geq 1$ to give a proof of Theorem 6.

Let X, Y be based spaces. We would like to study the relationship between the based and unbased homotopy classes of maps $[X, Y]_*$ and $[X, Y]$. This involves defining a right action of $\pi_1(Y)$ on the set $[X, Y]_*$ given by homotopy extensions. A first step is to introduce the following.

Definition 8.

- A continuous map $i : A \longrightarrow X$ is a *cofibration* (or Hurewicz cofibration) if given any solid commutative diagram of the form

(4)



  in Top, there exists a continuous map $\widehat{f}$ such that (4) commutes.
- Let X be a space and $A \subseteq X$ a subspace. The pair (X, A) has the *homotopy extension property* (HEP) if the natural inclusion map $i : A \longrightarrow X$ is a cofibration.
- A based space (X, x_0) is *well-pointed* (or non-degenerately based) if the natural inclusion $i : \{x_0\} \longrightarrow X$ is a cofibration and $\{x_0\} \subseteq X$ is closed.

We will use the homotopy extension property in the next exercise to prove the following.

Proposition 9. *Let (X, x_0) and (Y, y_0) be based spaces. If (X, x_0) is well-pointed, then there is a right action of $\pi_1(Y)$ on the set $[X, Y]_*$ given by homotopy extensions. If Y is path connected, then there is a bijection*

$$[X, Y]_* / \pi_1(Y, y_0) \cong [X, Y]$$

between the orbit set and the set of homotopy classes of maps $[X, Y]$.

The following is an immediate corollary.

Proposition 10. *Let (X, x_0) and (Y, y_0) be based spaces. If (X, x_0) is well-pointed and Y is simply connected, then there is a bijection*

$$[X, Y]_* \cong [X, Y]$$

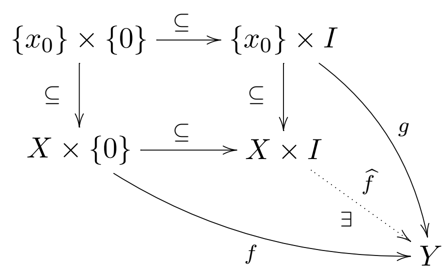
between based and unbased homotopy classes of maps.

Exercise 11. The purpose of this exercise is to prove Proposition 9. Let (X, x_0) and (Y, y_0) be based spaces. Suppose that (X, x_0) is well-pointed. A right action of $\pi_1(Y, y_0)$ on the set $[X, Y]_*$ is defined by homotopy extensions as follows:

$$(5) \qquad [X, Y]_* \times \pi_1(Y, y_0) \longrightarrow [X, Y]_*, \quad ([f], [g]) \longmapsto [f] \cdot [g] := [\widehat{f}(-, 1)].$$

Here, $\widehat{f}(-, 1) : X \longrightarrow Y$ is the restriction to $X \cong X \times \{1\}$ of any homotopy $\widehat{f} : X \times I \longrightarrow Y$ such that $\widehat{f}(-, 0) = f$ and $\widehat{f}(x_0, -) = g$; in other words, the map $\widehat{f}(-, 1) : X \longrightarrow Y$ is the restriction to $X \cong X \times \{1\}$ of any continuous map $\widehat{f} : X \times I \longrightarrow Y$ which makes the diagram

(6)



commute.

(a) Prove that (5) is a well-defined function.
(b) Prove that (5) defines a right $\pi_1(Y, y_0)$-action on $[X, Y]_*$.

Define a function Φ by forgetting basepoints as follows:

$$\Phi : [X, Y]_* \longrightarrow [X, Y], \quad [f] \longmapsto [f].$$

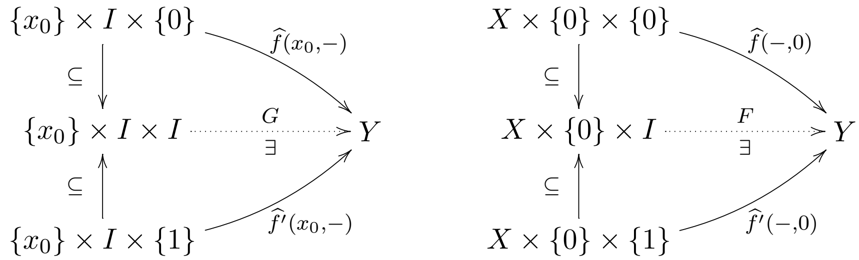
(c) Prove that Φ induces a well-defined function

$$\overline{\Phi} : [X, Y]_*/\pi_1(Y, y_0) \longrightarrow [X, Y]$$

between the orbit set and the set of homotopy classes of maps $[X, Y]$.

(d) Prove that $\overline{\Phi}$ is injective.
(e) Assume furthermore that Y is path connected. Prove that $\overline{\Phi}$ is surjective. This completes the proof of Proposition 9.

Part (a) can be argued as follows: consider maps $f, f' : (X, x_0) \longrightarrow (Y, y_0)$ such that $[f] = [f'] \in [X, Y]_*$ and loops $g, g' : (I, \partial I) \longrightarrow (Y, y_0)$ such that $[g] = [g'] \in \pi_1(Y, y_0)$. Consider any corresponding $\widehat{f}, \widehat{f}' : X \times I \longrightarrow Y$ as above. Since g and g' are path homotopic, there exists a path homotopy G which makes the left-hand diagram



commute. Since f and f' are based homotopic, there exists a based homotopy F which makes the right-hand diagram commute. Since $\{x_0\} \subseteq X$ is closed, the continuous maps

$$X \times I \times \{0\} \xrightarrow{\widehat{f}} Y, \quad X \times I \times \{1\} \xrightarrow{\widehat{f'}} Y, \quad X \times \{0\} \times I \xrightarrow{F} Y, \quad \{x_0\} \times I \times I \xrightarrow{G} Y$$

define a continuous map H of the form

$$\begin{array}{ccc} B := (X \times I \times \partial I) \cup (X \times \{0\} \times I) \cup (\{x_0\} \times I \times I) & \xrightarrow{H} & Y \\ \downarrow \subseteq & \nearrow \widehat{H} & \\ X \times I \times I & & \end{array}$$

To complete the proof, it is enough to verify there exists a continuous map $\widehat{H}$ which makes the diagram commute; to see this, note that restricting any such $\widehat{H}$ to $X \times \{1\} \times I$ gives a based homotopy from $\widehat{f}(-,1)$ to $\widehat{f'}(-,1)$. Hence to complete the proof, it is enough to verify that such an extension $\widehat{H}$ exists. Note that there is a homeomorphism of pairs

$$(I \times I, I \times \partial I \cup \{0\} \times I) \cong (I \times I, I \times \{0\}).$$

Hence to verify that such an $\widehat{H}$ exists, it is enough to verify there exists a continuous map $\widetilde{H}$ which makes the diagram

$$\begin{array}{ccccc} X \times I \times \{0\} \cup \{x_0\} \times I \times I & \xrightarrow{\cong} & B & \xrightarrow{H} & Y \\ \downarrow \subseteq & & & \nearrow \widetilde{H} & \\ X \times I \times I & & & & \end{array}$$

commute. Since (X, x_0) has the HEP, it follows that $(X \times I, \{x_0\} \times I)$ has the HEP, which implies that such an extension $\widetilde{H}$ exists. This completes the argument for part (a) that (5) is a well-defined function.

The following explains the name "cofibration" and will be used below in the exercises.

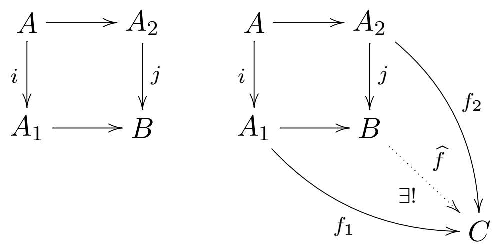
Exercise 12. Let $i : A \longrightarrow X$ be a continuous map. Prove that i is a cofibration if and only if given any solid commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & \mathrm{Map}(I, Y) \\ {\scriptstyle i}\downarrow & \overset{\widehat{f}}{\nearrow} \; \exists & \downarrow {\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow{f} & Y \end{array} \tag{7}$$

in Top, there exists a continuous map $\widehat{f}$ such that (7) commutes. Here, ev_0 is the "evaluate at 0" map.

Recall that a (left-hand side) commutative diagram in Top of the form

(8)



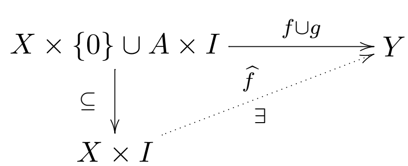
is a *pushout diagram* if is satisfies the universal property: given two continuous maps f_1, f_2 such that the outer (right-hand side) diagram commutes, then there exists a unique continuous map $\widehat{f}$ which makes the diagram commute.
The following two exercises will be used below to prove that $(S^n, *)$ is well-pointed.

Exercise 13. Consider any pushout diagram of the form (8). Use the lifting condition in Exercise 12 to prove the following: if i is a cofibration, then j is a cofibration.

Exercise 14. Let X be a space and $A \subseteq X$ a closed subspace.

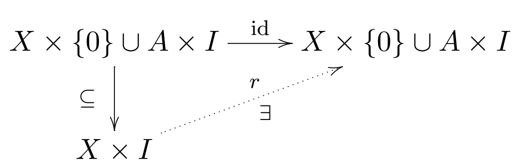
(a) Prove that (X, A) has the homotopy extension property if and only if given any solid diagram of the form

(9)



in Top, there exists a continuous map $\widehat{f}$ such that (9) commutes.

(b) Prove that (X, A) has the homotopy extension property if and only if there exists a continuous map r which makes the diagram



commute; i.e., if and only if the subspace $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Exercise 15. Let $n \geq 1$.

(a) Prove that $(D^n, \partial D^n)$ has the homotopy extension property.
(b) Prove that $(S^n, *)$ is well-pointed.
(c) Let Y be a based space. Conclude that $\pi_n(Y)$ has a right $\pi_1(Y)$-action given by homotopy extensions.

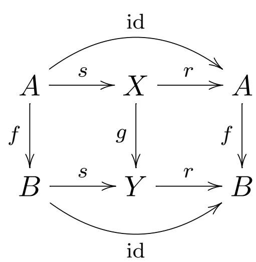
For part (a), show that $D^n \times \{0\} \cup \partial D^n \times I$ is a strong deformation retract of $D^n \times I$ by considering a radial projection. For part (b), use the following pushout diagram description of S^n

$$\begin{array}{ccc} \partial D^n & \longrightarrow & * \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & D^n/\partial D^n \cong S^n \end{array}$$

to prove that $* \longrightarrow S^n$ is a cofibration.

The following definition arises frequently in homotopy theory.

Definition 16. Let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ be continuous maps. Then f is a *retract* of g if there is a commutative diagram of the form



in Top.

Exercise 17. Use the lifting condition in Exercise 12 to prove the following:

(a) Every homeomorphism is a cofibration.
(b) The composition of two cofibrations is a cofibration.
(c) A retract of a cofibration is a cofibration.
(d) The disjoint union of two cofibrations is a cofibration.
(e) An arbitrary disjoint union of cofibrations is a cofibration.
(f) If $i : A \longrightarrow X$ is a cofibration, then $i \times \mathrm{id} : A \times I \longrightarrow X \times I$ is a cofibration.

Here are some references for this material: [1, Sections 4.1, 4.4], [2, Chapter VII], [3, Sections 1.1, 4.A], [4, Chapter 16].

References

[1] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic topology from a homotopical viewpoint*. Universitext. Springer-Verlag, New York, 2002. Translated from the Spanish by Stephen Bruce Sontz.

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[4] Czes Kosniowski. *A first course in algebraic topology*. Cambridge University Press, Great Britain, 1980.