Higher Algebraic $K$-theory
Exercise Set 10

06.12.2011

(Homological resolutions.) Let $(\mathcal{A}, \mathcal{E})$ be an exact category, and let $\mathcal{B}$ be an abelian category. A collection of functors $\mathcal{T} = \{T_n : \mathcal{A} \to \mathcal{B} \mid n \geq 1\}$ is homological if every exact sequence $E = (M' \to M \to M'')$ gives rise to a long exact sequence in $\mathcal{B}$

$$\cdots \to T_n(M) \to T_n(M'') \xrightarrow{\partial_n} T_{n-1}(M') \to \cdots \to T_1(M) \to T_1(M''),$$

where the connecting maps $\partial_E$ are natural in $E$. An object $M$ in $\mathcal{A}$ is then said to be $\mathcal{T}$-acyclic if $T_n(M) = 0$ for all $n \geq 0$.

1. Let $\mathcal{T}$ be a homological collection as above. Let $\mathcal{P}(\mathcal{T})$ denote the full subcategory of $\mathcal{T}$-acyclic objects in $\mathcal{A}$. Prove that for every exact sequence $M' \to M \to M''$,

$$M', M'' \in \mathcal{P}(\mathcal{T}) \implies M \in \mathcal{P}(\mathcal{T})$$

and

$$M, M'' \in \mathcal{P}(\mathcal{T}) \implies M' \in \mathcal{P}(\mathcal{T}).$$

2. Suppose that for all $M \in \mathcal{A}$, there exist a deflation $P \to M$ such that $P \in \mathcal{P}(\mathcal{T})$ and a natural number $n$ such that $T_n(M) = 0$ for all $k \geq 0$. Prove that the inclusion $\mathcal{P}(\mathcal{T}) \to \mathcal{A}$ induces isomorphisms

$$K_n(\mathcal{P}(\mathcal{T}), \mathcal{P}(\mathcal{T}) \cap \mathcal{E}) \xrightarrow{\cong} K_n(\mathcal{A}, \mathcal{E})$$

for all $n \geq 0$.

3. Interpret the result in 1(b) in each of the following situations.

(a) $\mathcal{A} = \mathcal{M}(A)$, the category of finitely generated modules over a ring $A$, and

$$\mathcal{T} = \{\text{Tor}_n^A(X, -) \mid n \geq 1\},$$

where $X$ is any finitely generated $A$-module of finite flat dimension.

(b) $\mathcal{A} = \text{Ch}^b(A)$, the category of bounded chain complexes of finitely generated $A$-modules, and

$$\mathcal{T} = \{H_n(-) \mid n \geq 1\}.$$