Higher Algebraic K-theory
Exercise Set 2

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1. The goal of this exercise is to prove the famous and very useful Yoneda Lemma.

   Notation 0.1. Given two functors $F, G : \mathscr{C} \to \mathscr{D}$, let $\mathsf{Nat}(F, G)$ denote the set of natural transformations from F to G.

   (a) Let $F : \mathscr{C} \to \mathbf{Set}$ be any functor. Prove that for all $C \in \operatorname{Ob}\mathscr{C}$ there is a bijection
   $$\mathsf{Nat}\big(\mathscr{C}(C, -), F\big) \cong F(C),$$
   which is natural in C and F. In particular, for all $C, C' \in \operatorname{Ob}\mathscr{C}$
   $$\mathsf{Nat}\big(\mathscr{C}(C, -), \mathscr{C}(C', -)\big) \cong \mathscr{C}(C', C),$$
   naturally in C and C'.

   (b) Let $G : \mathscr{C}^{op} \to \mathbf{Set}$ be any functor. Prove that for all $C \in \operatorname{Ob}\mathscr{C}$ there is a bijection
   $$\mathsf{Nat}\big(\mathscr{C}(-, C), G\big) \cong G(C),$$
   which is natural in C and G. In particular, for all $C, C' \in \operatorname{Ob}\mathscr{C}$
   $$\mathsf{Nat}\big(\mathscr{C}(-, C), \mathscr{C}(-, C')\big) \cong \mathscr{C}(C, C'),$$
   naturally in C and C'.

2. We now apply the Yoneda Lemma to describing simplicial sets and their morphisms

   (a) Let $K_\bullet$ be a simplicial set. Prove that $K_n \cong \mathbf{sSet}(\Delta[n], K_\bullet)$ for all n.

   (b) Prove that for all $m, n \geq 0$
   $$\mathbf{sSet}\big(\Delta[m], \Delta[n]\big) \cong \mathbf{\Delta}(m, n),$$
   naturally in m and n.

3. In this exercise we study the simplicial analog of the topological mapping space.

   (a) Let $K_\bullet$ and $L_\bullet$ be simplicial sets. Explain how to define the faces and degeneracies of a simplicial set $\mathrm{Map}(K_\bullet, L_\bullet)_\bullet$ with
   $$\mathrm{Map}(K_\bullet, L_\bullet)_n = \mathbf{sSet}(K_\bullet \times \Delta[n], L_\bullet),$$
   using the maps δ^i and σ^j.

   (b) Prove that for any simplicial set $K_\bullet$, the functors $-\times K_\bullet$ and $\mathrm{Map}(K_\bullet, -)$ are adjoint.

   (c) Use (b) and the Yoneda Lemma to prove that there is a natural isomorphism of simplicial sets
   $$\mathrm{Map}\left(J_\bullet, \mathrm{Map}(K_\bullet, L_\bullet)\right) \cong \mathrm{Map}(J_\bullet \times K_\bullet, L_\bullet),$$
   for all simplicial sets $J_\bullet, K_\bullet, L_\bullet$.

4. Show that for all $m, n \geq 0$ and $f \in \mathbf{\Delta}(m, n)$, there exist unique sets of integers
   $$n \geq 1_1 > \cdots > i_k \geq 0 \quad \text{and} \quad 0 \leq j_1 < \cdots < j_l \leq m$$
   such that
   $$f = \partial^{i_1} \cdots \partial^{i_k} \sigma^{j_1} \cdots \sigma^{j_l},$$
   where $n - k + l = m$.

5. Prove that $\mathbf{sSet}$ is complete and cocomplete.