Higher Algebraic K-theory
Exercise Set 3

11.10.2011

1. Prove the *Eilenberg-Zilber Lemma*: Let $K_\bullet$ be a simplicial set. For every $m \geq 0$ and every $x \in K_m$, there is an natural number $p \leq m$, a non-degenerate p-simplex y, and a sequence $0 \leq j_1 < \cdots < j_{m-p} < m$ such that $x = s_{j_{m-p}} \cdots s_{j_1} y$. Moreover, the data $(p; y; j_1, ..., j_{m-p})$ are uniquely determined by x.

2. The goal of this exercise is to introduce and study the elementary properties of the *nerve functor*, $\mathsf{N}_\bullet : \mathbf{Cat} \to \mathbf{sSet}$.

   (a) Let **Poset** denote the category of posets (partially ordered sets), and let **Cat** denote the category of small categories. Define a functor $\iota : \mathbf{Poset} \to \mathbf{Cat}$ such that $\mathrm{Ob}\,\iota(P, <) = P$ and that is *faithful*, i.e., injective on morphisms. The functor ι allows us to view any poset, such as the totally ordered set $[n]$, as a category in a natural way.

   (b) Let $\mathsf{N}_\bullet : \mathbf{Cat} \to \mathbf{sSet}$ be the functor defined by
   $$\mathsf{N}_\bullet \mathscr{C} = \mathbf{Cat}\big(\iota(-), \mathscr{C}\big) : \mathbf{\Delta}^{op} \to \mathbf{Set}.$$
   Show that $\mathsf{N}_0 \mathscr{C} = \mathrm{Ob}\,\mathscr{C}$, while for all $n > 0$,
   $$\mathsf{N}_n \mathscr{C} = \big\{ C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n \mid f_i \in \mathrm{Mor}\,\mathscr{C}\ \forall i \big\}.$$
   Describe explicitly the face maps $d_i : \mathsf{N}_n \mathscr{C} \to \mathsf{N}_{n-1} \mathscr{C}$, the degeneracies $s_j : \mathsf{N}_n \mathscr{C} \to \mathsf{N}_{n+1} \mathscr{C}$, and the simplicial map $\mathsf{N}_\bullet F : \mathsf{N}_\bullet \mathscr{C} \to \mathsf{N}_\bullet \mathscr{D}$ induced by a functor $F : \mathscr{C} \to \mathscr{D}$. What are the nondegenerate simplices of $\mathsf{N}_\bullet \mathscr{C}$?

   (c) Show that $\mathsf{N}_\bullet \iota[n] = \Delta[n]$ for all $n \geq 0$.

   (d) There is a functor $\Lambda : \mathbf{Gr} \to \mathbf{Cat}$ such that $\mathrm{Ob}\,\Lambda(G) = \star$ and $\Lambda(G)(\star, \star) = G$, where composition is given by multiplication in G. The composite functor $\mathsf{B}_\bullet = \mathsf{N}_\bullet \circ \Lambda$ is the *simplicial bar construction.* Calculate $\mathsf{B}_\bullet(\mathbb{Z}/2\mathbb{Z})$. What are its nondegenerate simplices?

   (e) Apply the Yoneda Lemma to proving that the nerve functor induces a bijection
   $$\mathsf{N}_\bullet : \mathbf{Cat}(\mathscr{C}, \mathscr{D}) \to \mathbf{sSet}(\mathsf{N}_\bullet \mathscr{C}, \mathsf{N}_\bullet \mathscr{D})$$
   for all small categories $\mathscr{C}$, $\mathscr{D}$.

(f) Prove that the nerve functor preserves all (small) limits and coproducts. (To show that $\mathsf{N}_\bullet$ preserves limits, one can construct a left adjoint to $\mathsf{N}_\bullet$, thereby establishing that $\mathsf{N}_\bullet$ is a right adjoint and therefore preserves limits. It is also possible simply to prove limit preservation directly.)