Higher Algebraic K-theory
Exercise Set 8

22.11.2011

1. (A generalization of the Additivity Theorem.) Let $(\mathscr{A}, \mathscr{E})$ be an exact category.

   (a) Let $\mathscr{D}$ be any small category. Show that the functor category $\mathscr{A}^{\mathscr{D}}$ inherits the structure of an exact category, in which a sequence of functors $F' \to F \to F''$ is exact if and only if $F'(D) \to F(D) \to F''(D)$ is exact for all $D \in \operatorname{Ob} \mathscr{D}$.

   (b) An *admissible filtration* of a functor $F : \mathscr{D} \to \mathscr{A}$ consists of a sequence of natural transformations

   $$0 = F_0 \xrightarrow{\iota_1} F_1 \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_{n-1}} F_{n-1} \xrightarrow{\iota_n} F_n = F \tag{1}$$

   of functors from $\mathscr{D}$ to $\mathscr{A}$ such that $(\iota_k)_D : F_{k-1}(D) \to F_k(D)$ is an inflation for all k and all D. Prove that any admissible filtration induces quotient functors $F_k/F_l : \mathscr{D} \to \mathscr{A}$ for all $k \geq l$, i.e., $F_k/F_l(D) = F_k(D)/F_l(D)$ for every object D.
   (Remark: If $i : A \rightarrowtail B$ is an inflation, then B/A denotes the pushout of $0 \xleftarrow{0} A \overset{i}{\rightarrowtail} B$.)

   (c) Prove that if $(\mathscr{D}, \mathscr{E}')$ is an exact category, and (1) is an admissible filtration of a functor $F : \mathscr{D} \to \mathscr{A}$ such that the functors F_k/F_{k-1} are exact for all k, then F_k/F_l is exact for all $k \geq l$.

   (d) (Additivity for *characteristic filtrations*) Let $F : (\mathscr{D}, \mathscr{E}') \to (\mathscr{A}, \mathscr{E})$ be an exact functor equipped with an admissible filtration (1) such tht F_k/F_{k-1} is exact for all k. Prove that

   $$K_m(F) = \sum_{k=1}^{n} K_m(F_k/F_{k-1})$$

   for all $m \geq 0$.

2. (Consequences of the Additivity Theorem and its generalization.)

   (a) Let R be any ring. Prove that $K_n(R)$ is naturally a $K_0(R)$-module for all n.

   Hint: If P is a projective R-module, then $P \otimes_R - : \mathscr{P}(R) \to \mathscr{P}(R)$ is an exact functor and thus induces a homomorphism

   $$\varphi_P = K_n(P \otimes_R -) : K_n(R) \to K_n(R).$$

   (b) Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring, e.g., the ring $\Bbbk[x]$ of polynomials in one variable x of degree 2 over a field $\Bbbk$. Let $\mathscr{P}_*(R)$ denote the category of $\mathbb{Z}$-graded R-modules that are finitely generated and projective, and let $\mathscr{E}_*$ denote the usual class of exact sequences of graded modules.

      i. Prove that the shift functor $\Sigma : \mathscr{P}_*(R) \to \mathscr{P}_*(R)$, given by $\Sigma(P)_n = P_{n-1}$ for all n, induces a $\mathbb{Z}[t, t^{-1}]$-module structure on $K_n\big(\mathscr{P}_*(R), \mathscr{E}_*\big)$.

      ii. Prove that there is an isomorphism of $\mathbb{Z}[t, t^{-1}]$-modules specified by

      $$\mathbb{Z}[t, t^{-1}] \otimes K_n(R_0) \to K_n\big(\mathscr{P}_*(R), \mathscr{E}_*\big) : 1 \otimes y \mapsto K_n(R \otimes_{R_0} -)(y).$$

      Hint: Define an admissible filtration of the identity functor on $\mathscr{P}_*(R)$ by letting $F_k(P)$ be the R-submodule of P generated by $\bigoplus_{n \leq k} P_n$ for every $P \in \operatorname{Ob} \mathscr{P}_*(R)$. Apply the isomorphism

      $$P \cong \coprod_{n \in \mathbb{Z}} \Sigma^n(R) \otimes_{R_0} (R_0 \otimes_R P)_n$$

      (due to Bass) to proving that all of the F_k and F_k/F_{k-1} are exact, then apply Additivity for characteristic filtrations.