Higher Algebraic $K$-theory
Exercise Set 8

22.11.2011

1. (A generalization of the Additivity Theorem.) Let $(\mathcal{A}, \mathcal{E})$ be an exact category.

(a) Let $\mathcal{D}$ be any small category. Show that the functor category $\mathcal{A}^\mathcal{D}$ inherits the structure of an exact category, in which a sequence of functors $F' \to F \to F''$ is exact if and only if $F'(D) \to F(D) \to F''(D)$ is exact for all $D \in \text{Ob} \mathcal{D}$.

(b) An admissible filtration of a functor $F : \mathcal{D} \to \mathcal{A}$ consists of a sequence of natural transformations

\[ 0 = F_0 \xrightarrow{\iota_1} F_1 \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_{n-1}} F_{n-1} \xrightarrow{\iota_n} F_n = F \] (1)

of functors from $\mathcal{D}$ to $\mathcal{A}$ such that $(\iota_k)_D : F_{k-1}(D) \to F_k(D)$ is an inflation for all $k$ and all $D$. Prove that any admissible filtration induces quotient functors $F_k/F_{k-1} : \mathcal{D} \to \mathcal{A}$ for all $k \geq l$, i.e., $F_k/F_l(D) = F_k(D)/F_l(D)$ for every object $D$.

(Remark: If $i : A \xrightarrow{i} B$ is an inflation, then $B/A$ denotes the pushout of $0 \xrightarrow{0} A \xrightarrow{i} B$.)

(c) Prove that if $(\mathcal{D}, \mathcal{E}')$ is an exact category, and (1) is an admissible filtration of a functor $F : \mathcal{D} \to \mathcal{A}$ such that the functors $F_k/F_{k-1}$ are exact for all $k$, then $F_k/F_l$ is exact for all $k \geq l$.

(d) (Additivity for characteristic filtrations) Let $F : (\mathcal{D}, \mathcal{E}') \to (\mathcal{A}, \mathcal{E})$ be an exact functor equipped with an admissible filtration (1) such that $F_k/F_{k-1}$ is exact for all $k$. Prove that

\[ K_m(F) = \sum_{k=1}^{n} K_m(F_k/F_{k-1}) \]

for all $m \geq 0$. 

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2. (Consequences of the Additivity Theorem and its generalization.)

(a) Let $R$ be any ring. Prove that $K_n(R)$ is naturally a $K_0(R)$-module for all $n$.

**Hint:** If $P$ is a projective $R$-module, then $P \otimes_R - : \mathcal{P}(R) \to \mathcal{P}(R)$ is an exact functor and thus induces a homomorphism

$$\varphi_P = K_n(P \otimes_R -) : K_n(R) \to K_n(R).$$

(b) Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring, e.g., the ring $\mathbb{k}[x]$ of polynomials in one variable $x$ of degree 2 over a field $\mathbb{k}$. Let $\mathcal{P}_*(R)$ denote the category of $\mathbb{Z}$-graded $R$-modules that are finitely generated and projective, and let $\mathcal{E}_*$ denote the usual class of exact sequences of graded modules.

i. Prove that the shift functor $\Sigma : \mathcal{P}_*(R) \to \mathcal{P}_*(R)$, given by $\Sigma(P)_n = P_{n-1}$ for all $n$, induces a $\mathbb{Z}[t,t^{-1}]$-module structure on $K_n(\mathcal{P}_*(R), \mathcal{E}_*)$.

ii. Prove that there is an isomorphism of $\mathbb{Z}[t,t^{-1}]$-modules specified by

$$\mathbb{Z}[t,t^{-1}] \otimes K_n(R_0) \to K_n(\mathcal{P}_*(R), \mathcal{E}_*) : 1 \otimes y \mapsto K_n(R \otimes R_0 -)(y).$$

**Hint:** Define an admissible filtration of the identity functor on $\mathcal{P}_*(R)$ by letting $F_k(P)$ be the $R$-submodule of $P$ generated by $\bigoplus_{n \leq k} P_n$ for every $P \in \text{Ob } \mathcal{P}_*(R)$. Apply the isomorphism

$$P \cong \prod_{n \in \mathbb{Z}} \Sigma^n(R) \otimes_{R_0} (R_0 \otimes_R P)_n$$

(due to Bass) to proving that all of the $F_k$ and $F_k/F_{k-1}$ are exact, then apply Additivity for characteristic filtrations.