Higher Algebraic K-theory
Exercise Set 9

29.11.2011

1. (Transfer maps.) Let $\varphi : R \to S$ be a ring homomorphism. Suppose that the induced left R-module structure on S is such that $S \in \mathscr{P}_\infty(R)$, i.e., S admits an R-projective resolution of finite length.

   (a) Prove that the "restriction of scalars" functor $\varphi^* : {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$ restricts and corestricts to an exact functor $\varphi^* : \mathscr{P}_\infty(S) \to \mathscr{P}_\infty(R)$.

   (b) Explain why the Resolution Theorem allows us to conclude that restriction of scalars induces a homomomorphism

   $$\mathrm{tr}(\varphi)_n : K_n(S) \to K_n(R),$$

   called the *transfer map* of φ, for all $n \geq 0$.

   (c) Let $\psi : S \to T$ be a ring homomorphism such that $T \in \mathscr{P}_\infty(S)$. Prove that $T \in \mathscr{P}_\infty(R)$ and that

   $$\mathrm{tr}(\psi \circ \varphi)_n = \mathrm{tr}(\varphi)_n \circ \mathrm{tr}(\psi)_n : K_n(T) \to K_n(R)$$

   for all $n \geq 0$.

   (d) Apply the Resolution Theorem to showing that extension of scalars $S \otimes_R - : {}_R\mathbf{Mod} \to {}_S\mathbf{Mod}$ also induces homomorphims

   $$K_n(\varphi) : K_n(R) \to K_n(S)$$

   for all $n \geq 0$.

   (e) (The projection formula) Suppose that R and S are commutative. Prove that

   $$\mathrm{tr}(\varphi)_n\big(K_0(\varphi)(x) \cdot y\big) = x \cdot \mathrm{tr}(\varphi)_n(y)$$

   for all $x \in K_0(R)$ and $y \in K_n(S)$, where $\cdot$ denotes the action either of $K_0(R)$ on $K_n(R)$ or of $K_0(S)$ on $K_n(S)$ (cf. Exercise 2(a) of Exercise Set 8).

   Hint: Note that since R and S are commutative, there is a natural isomorphisms $(S \otimes_R M) \otimes_S N \cong M \otimes_R \varphi^*(N)$ for all R-modules M and S-modules N.

2. (A useful technical lemma.) Let $(\mathscr{A}, \mathscr{E})$ be an exact category, and let $\mathscr{P}$ be a full subcategory of $\mathscr{A}$ such that if $M' \rightarrowtail M \twoheadrightarrow M''$ is an exact sequence, then
$$M', M'' \in \mathscr{P} \Longrightarrow M \in \mathscr{P}$$
and
$$M, M'' \in \mathscr{P} \Longrightarrow M' \in \mathscr{P}.$$
Prove that for every exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$ and for every $n \geq 0$,
$$M' \in \mathscr{P}_n, M'' \in \mathscr{P}_{n+1} \Longrightarrow M' \in \mathscr{P}_n,$$
$$M', M'' \in \mathscr{P}_{n+1} \Longrightarrow M \in \mathscr{P}_{n+1},$$
and
$$M, M'' \in \mathscr{P}_{n+1} \Longrightarrow M \in \mathscr{P}_{n+1}.$$