Exercises 1, March 10, 2006

Categories

1) Let $\mathscr{C}$ be a category and let $\mathscr{C}^{op}$ be the the *opposite category of* $\mathscr{C}$ defined by $\mathrm{Ob}\mathscr{C}^{op} = \mathrm{Ob}\mathscr{C}$ and $\mathscr{C}^{op}(X,Y) = \mathscr{C}(Y,X)$.

   If $f^{op} \in \mathscr{C}^{op}(X,Y)$ and $g^{op} \in \mathscr{C}^{op}(Y,Z)$ then composition of morphisms is defined by the formula

$$f^{op} \circ g^{op} = (g \circ f)^{op}$$

   or, equivalently, by the following commutative diagram:

$$\begin{array}{ccc} \mathscr{C}^{op}(X,Y) \times \mathscr{C}^{op}(Y,Z) & \xrightarrow{\circ} & \mathscr{C}^{op}(X,Z) \\ \| & & \| \\ \mathscr{C}(Y,X) \times \mathscr{C}(Z,Y) & & \| \\ \downarrow \cong & & \| \\ \mathscr{C}(Z,Y) \times \mathscr{C}(Y,X) & \longrightarrow & \mathscr{C}(Z,X)\,. \end{array}$$

   Verify that $\mathscr{C}^{op}$ is a category.

2) Let $\mathscr{I}$ and $\mathscr{C}$ be two categories. Let $\mathscr{C}^{\mathscr{I}}$ be the category with objects the collection of functors $\mathscr{I} \to \mathscr{C}$. Given two functors $F, G : \mathscr{I} \to \mathscr{C}$. Then the morphism set $\mathscr{C}^{\mathscr{I}}(F,G)$ is the set of natural transformations from F to G.

   Verify that $\mathscr{C}^{\mathscr{I}}$ is a category.

3 a) Let $\mathscr{I}$ be a *small* category, i.e., the objects $\mathrm{Ob}\mathscr{I}$ form a set and let $\mathscr{C}$ be any category. A functor $D : \mathscr{I} \to \mathscr{C}$ is called an *$\mathscr{I}$-shaped diagram in $\mathscr{C}$.*

   Let $Z \in \mathrm{ob}\mathscr{C}$ and assume that the diagram D maps to Z. This means that for all $A, B \in \mathrm{Ob}\mathscr{I}$ and for all $\phi \in \mathscr{I}(A,B)$ you have maps f_A and f_B such that the following diagram commutes:

$$\begin{array}{ccc} D(A) & \xrightarrow{f_A} & Z\,. \\ {\scriptstyle D(\phi)}\downarrow & \nearrow_{f_B} & \\ D(B) & & \end{array}$$

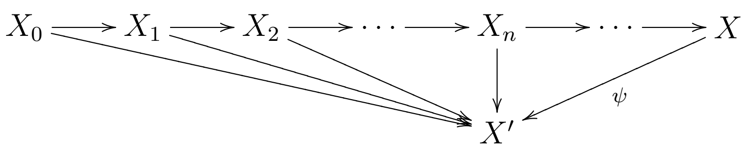
Let now $\mathscr{I}$ be the category with objects equal to the set of non-negative integers $\{0, 1, 2, \ldots, n, \ldots\}$. There is a unique morphism from n to m if and only if $n \leq m$.

A functor $D : \mathscr{I} \to \mathscr{C}$, or an $\mathscr{I}$-shaped diagram in a category $\mathscr{C}$, is now a sequential limit system, which can be identified with the diagram

$$X_0 \to X_1 \to X_2 \to \ldots \to X_n \to \ldots$$

where the objects and morphisms live in $\mathscr{C}$.

Assume that there exists an object $X \in \mathscr{C}$ and a map from the above diagram to X such that the following holds: Every time you have an object X' and a map from the diagram to X', then there exists a *unique* map $\psi : X \to X'$ such that the following diagram commutes:



When such an object exists, we call it the *colimit* of the functor D (or the diagram) and denote it by $\operatorname{colim}_n X_n$.

Assume that $\operatorname{colim}_n X_n$ exists. Show that this object is unique up to isomorphism.

3 b) Find the colimit of the constant sequential limit system

$$X = X = X = \cdots = X = \cdots .$$

3 c) Given two sequential limit systems and a set of maps between them. Precisely, we have a commutative diagram in the category $\mathscr{C}$

$$\begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow \cdots \longrightarrow & X_n & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow \cdots \longrightarrow & Y_n & \longrightarrow & \cdots . \end{array}$$

Assume that both (horizontal) colimits exist. Show that this diagram defines a morphism $\operatorname{colim}_n X_n \to \operatorname{colim}_n Y_n$.

Moreover, assume that each vertical map in the diagram above is an isomorphism. Show that the induced map of colimit is also an isomorphism.

3 d) Let $\mathscr{C} = Ab$ be the category of Abelian groups. I.e. the objects are Abelian groups and the morphisms are homomorphisms of groups.

Compute the colimit of the sequential system

$$\mathbb{Q} \xrightarrow{\cdot r} \mathbb{Q} \xrightarrow{\cdot r} \mathbb{Q} \xrightarrow{\cdot r} \cdots \xrightarrow{\cdot r} \mathbb{Q} \xrightarrow{\cdot r} \cdots$$

where every object is the additive group of rational numbers and the morphisms are given by multiplication with a fixed rational number r.

(Hint: Use 3 b) and 3 c).)