Exercises 13, June 20, 2006

The Künneth formula in cohomology

1) Let $X_\bullet$ and $Y_\bullet$ be simplicial sets, and let $UC_*(-)$ be the functor that to a simplicial set assigns the unreduced chain complex.

   Recall that $UC_*(X_\bullet)$ is then in degree n the free Abelian group generated by *all* the n-simplices of $X_\bullet$.

   Show by calculation that the following square commutes:

$$\begin{array}{ccc}
UC_*(X_\bullet \times Y_\bullet) & \xrightarrow{UC_*(\Delta)} & UC_*(X_\bullet \times Y_\bullet \times X_\bullet \times Y_\bullet) \\
 & & \downarrow{\scriptstyle C_*(1_X \times \tau_{YX} \times 1_Y)} \\
\downarrow{\scriptstyle f} & & UC_*(X_\bullet \times X_\bullet \times Y_\bullet \times Y_\bullet) \\
 & & \downarrow{\scriptstyle f} \\
UC_*(X_\bullet) \otimes UC_*(Y_\bullet) & \xrightarrow{UC_*(\Delta)\otimes UC_*(\Delta)} & UC_*(X_\bullet \times X_\bullet) \otimes UC_*(Y_\bullet \times Y_\bullet)
\end{array}$$

2) Show that $H_k(\Delta[n]_\bullet) = 0$ when $k > 0$.

   One way of doing this is to use that the geometric realization of $\Delta[n]_\bullet$ is homeomorphic to the topological n-simplex Δ^n. You might convince yourself that this is true by checking by hand in the case $n = 1, 2$ or even $n = 3$.

   Use the Künneth formula in homology to show that $H_i(\Delta[m]_\bullet \times \Delta[n]_\bullet \times \Delta[m]_\bullet \times \Delta[n]_\bullet) = 0$ for all $m, n \geq 0$ and $i > 0$.

3) Let $F, G : \underline{sSet} \times \underline{sSet} \to \underline{Ch}$ be two functors that to a pair of simplicial sets $(X_\bullet, Y_\bullet)$ assigns the following chain complexes:

$$\begin{aligned}
F(X_\bullet, Y_\bullet) &= UC_*(X_\bullet \times Y_\bullet \times X_\bullet \times Y_\bullet) \\
G(X_\bullet, Y_\bullet) &= UC_*(X_\bullet) \otimes UC_*(Y_\bullet) \otimes UC_*(X_\bullet) \otimes UC_*(Y_\bullet)
\end{aligned} \qquad (1)$$

   Redo exercises 3 b) and c) of last week's exercise, for these two functors.

4) Show, using the previous exercise, that the following diagram commutes up to a natural chain homotopy:

$$\begin{array}{ccc}
UC_*(X_\bullet \times Y_\bullet \times X_\bullet \times Y_\bullet) & \xrightarrow{\quad f \quad} & UC_*(X_\bullet \times Y_\bullet) \otimes UC_*(X_\bullet \times Y_\bullet) \\
\Big\downarrow {\scriptstyle UC_*(1\times\tau\times 1)} & & \Big\downarrow {\scriptstyle f\otimes f} \\
UC_*(X_\bullet \times X_\bullet \times Y_\bullet \times Y_\bullet) & & UC_*(X_\bullet) \otimes UC_*(Y_\bullet) \otimes UC_*(X_\bullet) \otimes UC_*(Y_\bullet) \\
\Big\downarrow {\scriptstyle f} & & {\scriptstyle 1\otimes T\otimes 1} \Big\uparrow \\
UC_*(X_\bullet \times X_\bullet) \otimes UC_*(Y_\bullet \times Y_\bullet) & \xrightarrow{f\otimes f} & UC_*(X_\bullet) \otimes UC_*(X_\bullet) \otimes UC_*(Y_\bullet) \otimes UC_*(Y_\bullet)
\end{array}$$

5) For a commutative and unital ring R, let $C^n(X_\bullet; R) = Hom(C_n(X_\bullet), R)$ denote the cochains of $X_\bullet$ with coefficients in R of degree n.

The map in the cohomological Künneth formula is defined on cochains by the Alexander-Whitney map f

$$C^m(X_\bullet; R) \otimes C^n(Y_\bullet; R) \to C^{m+n}(X_\bullet \times Y_\bullet; R) \qquad (2)$$

by the formula

$$\phi \otimes \psi \mapsto \mu_R \circ (\phi \otimes \psi) \circ f\,.$$

Here $\mu_R : R \otimes R \to R$ is the multiplication in the coefficient ring.

Remember that for a simplicial set $Z_\bullet$, the cup product is given as follows: Let $\alpha, \beta \in C^*(Z_\bullet; R) = Hom(C_*(Z_\bullet), R)$ be cochains of degree m and n, respectively. Then $\alpha \cup \beta : C_{m+n}(Z_\bullet) \to R$ is defined by the composite

$$C_*(Z_\bullet) \to C_*(Z_\bullet \times Z_\bullet) \stackrel{f}{\to} C_*(Z_\bullet) \otimes C_*(Z_\bullet) \stackrel{\alpha\otimes\beta}{\to} R \otimes R \stackrel{\mu_R}{\to} R\,.$$

Show, using 2) and 4), that the map (2) induces a homomorphism of algebras upon passing to cohomology.

6) Let $S^n_\bullet$ be the simplicial n-sphere, given by the quotient simplicial set $\Delta[n]_\bullet/\partial\Delta[n]_\bullet$.

Write down the reduced chain complex $C_*(S^n_\bullet)$ and calculate $H_*(S^m_\bullet \times S^n_\bullet)$ using the Künneth formula.

Use the Universal coefficient theorem to calculate the cohomology groups $H^*(S^m_\bullet \times S^n_\bullet; R)$ for any ring R.

7) Let k be a field. Then the cohomological Künneth formula says that

$$H^*(X_\bullet; k) \otimes H^*(Y_\bullet; k) \to H^*(X_\bullet \times Y_\bullet; k)$$

is an isomorphism. Use this to describe the algebra structure of the cohomology ring $H^*(S^m_\bullet \times S^n_\bullet; k)$.

8) Show that $S^m \times S^n$ is not homotopy equivalent to $W := S^n \vee S^m \vee S^{m+n}$ by comparing the product structures in cohomology with coefficients in a field k.

To do this, show directly using the definition that the cup product $H^n(W; k) \otimes H^m(W; k) \to H^{m+n}(W; k)$ is trivial.