Exercises 2, March 22, 2006

Chain complexes

1) Let Ch be the category of non-negatively graded chain complexes of Abelian groups. Show that the assignment

$$(C_*, d_*) \mapsto \{H_n(C_*, d_*)\}_{n \geq 0}$$

is a functor from Ch to the category of graded Abelian groups.

2) Let (C_*, d_*) be a chain complex of Abelian groups with differential $d_n : C_n \to C_{n-1}$ for all $n \geq 1$. Let M be a fixed Abelian group.

Consider the complex gotten by applying the functor $\mathrm{Hom}(-, M)$. I.e. for each $n \geq 0$, we have $C^n = \mathrm{Hom}(C_n, M)$ and a map

$$d^n : C^n \to C^{n+1}\,.$$

Show that (C^*, d^*) is a cochain complex.

3) Consider the following morphism of chain complexes:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\cdot p} & \mathbb{Z} & \xrightarrow{0} & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \xrightarrow{=} & 0 & \xrightarrow{=} & 0 & \xrightarrow{=} & 0 & \xrightarrow{=} & 0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & 0
\end{array} \tag{1}$$

Calculate the homology of both (horizontal) chain complexes, and show that the map induced in homology by the vertical maps is an isomorphism.

4) Let (C_*, d_*) and (C'_*, d'_*) be chain complexes of Abelian groups. Let $\mathrm{Hom}((C_*, d_*), (C'_*, d'_*))$ be the cochain complex which in degree n is the Abelian group consisting of sequences $\{f_k\}_k$, where $f_k \in \mathrm{Hom}(C_k, C'_{k-n})$ for every k.

The differential $\delta^n(\{f_k\}_k) = \{\phi_k\}_k$ is given by the formula

$$\phi_k = d'_{k-n} \circ f_k - (-1)^n f_{k-1} \circ d_k\,.$$

Show that $(\mathrm{Hom}((C_*, d_*), (C'_*, d'_*)), \delta^*)$ is a cochain complex.

5 a) Let $f : H \to H'$ be a homomorphism of Abelian groups and let $F_* \to H \to 0$ and $F'_* \to H' \to 0$ be two free resolutions of H and H'.

Show that there exists homomorphisms $\phi_n : F_n \to F'_n$ for all n such that the following diagram commutes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & F'_2 & \xrightarrow{d'_2} & F'_1 & \xrightarrow{d'_1} & F'_0 & \xrightarrow{d'_0} & H' & \longrightarrow & 0\,.
\end{array}$$

We say that the ϕ_n's above *extend* the homomorphism f.

5 b) Assume that $\{\psi_n : F_n \to F'_n\}_n$ is another sequence of homomorphisms extending f. Let $\beta_n := \psi_n - \phi_n$ for each n. Then these homomorphisms extends the zero homomorphism $H \xrightarrow{0} H'$. Show that there exists a chain homotopy between β and the zero homomorphism.

Precisely, show by induction that there exists maps

$$\lambda_n : F_n \to F'_{n+1}$$

such that

$$\beta_n = d'_{n+1} \circ \lambda_n + \lambda_{n-1} \circ d_n$$

5 c) Deduce from the two previous exercises that any two free resolutions of an Abelian group are chain homotopy equivalent to each other.

6 a) Consider the following map of chain complexes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \| & & \| & & \downarrow & & \| & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/3 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Show that the vertical maps induce the trivial map on homology.

6 b) Show that the map in the previous exercise is not chain homotopic to the zero map.