

Exercises 3, March 28, 2006

Chain complexes II

1 a) Let p be a fixed prime and let

$$\dots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow 0 \rightarrow \dots$$

$$\dots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \rightarrow \dots$$

be two chain complexes. Calculate the homology of each complex.

1 b) Apply the functor $\text{hom}(-, \mathbb{Z}/p)$ to the two chain complexes.

Calculate the cohomology of the two resulting cochain complexes.

2 a) Let

$$\dots \rightarrow 0 \rightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \rightarrow 0 \rightarrow \dots$$

$$\dots \rightarrow 0 \rightarrow \mathbb{Z}/3 \xrightarrow{2} \mathbb{Z}/6 \rightarrow 0 \rightarrow \dots$$

be two chain complexes.

Calculate the homology of each complex.

2 b) Assume that (C_*, d_*) and (C'_*, d'_*) are two chain complexes with isomorphic homology groups.

Is it true in general that we can find a map of chain complexes $f : C_* \rightarrow C'_*$ such that $H_*(f)$ is an isomorphism?

3 a) Assume that $(C[k]_*, d[k]_*)$ is a chain complex for every $k \geq 0$, and that we have inclusions of chain complexes

$$C[0]_* \dots \subset C[k-1]_* \subset C[k]_* \subset C[k+1]_* \subset \dots$$

I.e. for every $k \geq 0$ and every n , we have a commuting diagram

$$\begin{array}{ccc} C[k]_n & \xhookrightarrow{i} & C[k+1]_n \\ \downarrow d[k]_n & & \downarrow d[k+1]_n \\ C[k]_{n-1} & \xhookrightarrow{i} & C[k+1]_{n-1} \end{array}$$

Let $C[\infty]_*$ be the degree wise colimit with respect to the inclusions. That is, for each n we have $C[\infty]_n = \operatorname{colim}_{k \rightarrow \infty} C[k]_n \cong \bigcup_{k \geq 0} C[k]_n$.

Show that $C[\infty]_*$ has differentials $d[\infty]_*$ and becomes a chain complex such that each inclusion $C[k]_* \hookrightarrow C[\infty]_*$ is a morphism of chain complexes.

3 b) The inclusions i above induces maps of homology groups

$$\begin{aligned} H_*(C[0]_*, d[0]_*) &\rightarrow \dots \\ &\rightarrow H_*(C[k]_*, d[k]_*) \rightarrow H_*(C[k+1]_*, d[k+1]_*) \rightarrow \dots \end{aligned}$$

Show that $\operatorname{colim}_{k \rightarrow \infty} H_*(C[k]_*, d[k]_*)$ is isomorphic to $H_*(C[\infty]_*, d[\infty]_*)$.

4) Finish the proof that short exact sequences of chain complexes induce long exact sequences in homology.