Exercises 9, May 15, 2006

Simplicial homotopy II

1 a) Let X and Y be topological spaces. Remember that we defined the *singular singular complex of* X to be the simplicial set $S_\bullet(X)$ with n-simplices $S_n(X) = \{f : \Delta^n \to X \mid f \text{ continuous}\}$ and face- and degeneracy maps induced by precomposition with the maps $\delta^i : \Delta^{n-1} \to \Delta^n$ and $\sigma^j : \Delta^{n+1} \to \Delta^n$ for $0 \le i, j \le n$.

Show that there is a canonical isomorphism

$$S_\bullet(X \times Y) \xrightarrow{\cong} S_\bullet(X) \times S_\bullet(Y)$$

of simplicial sets.

1 b) Recall that the topological n-simplex Δ^n was defined as the the subspace

$$\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \le t_i \le 1, \sum_{i=0}^{n} t_i = 1\}.$$

Furthermore we have a functor $R : \underline{\Delta} \to \underline{Top}$, sending $(n) \in ob\ \underline{\Delta}^n$ to Δ^n and a morphism $\phi : (m) \to (n)$ to the map $\Delta^m \to \Delta^n$ defined by sending the ith vertex in Δ^m to the $\phi(i)$th vertex in Δ^n and extended linearly over Δ^m.

For instance, $R(\delta^i) : \Delta^{n-1} \hookrightarrow \Delta^n$ is the inclusion of the $(n-1)$ face that does not contain the ith vertex of Δ^n. Explicitly, $R(\delta^i)$ is the inclusion

$$(t_0, t_1, \ldots, t_{n-1}) \mapsto (t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

that also we denote by δ^i.

Show that there is a map $\Delta[n]_\bullet \to S_\bullet(\Delta^n)$ of simplicial sets.

Homology

2 a) Let G be a group with unit element $e \in G$. Define the classifying simplicial set of G, denoted $BG_\bullet$, by letting the n-simplices be

$$BG_n = \{(g_1, g_2, \ldots, g_n) \mid g_i \in G\} = G^{\times n}$$

for $n \ge 1$ and $BG_0 = \{()\}$ be the singleton set.

The degeneracy maps are defined as follows. For $n \geq 0$ and $0 \leq j \leq n$, we let

$$s_j(g_1, \dots, g_n) = (g_1, \dots, g_j, e, g_{j+1}, \dots, g_n)\,.$$

For $n > 0$ the face maps are defined by

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & i = 0 \\ (g_1, \dots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2}, \cdots g_n) & 0 < i < n \\ (g_1, \dots, g_{n-1}) & i = n \end{cases}.$$

Verify that $BG_\bullet$ is a simplicial set, i.e. verify the simplicial identities.

Remark: The simplicial set described above in fact *the nerve of a small category.* Let $\mathbb{G}$ be the category with a single object $*$ and morphisms $\mathbb{G}(*,*) = G$ equal the group G. Composition of morphisms is multiplication in the group.

The nerve of $\mathbb{G}$ is the simplicial set with n-simplices equal to the set of chains of composable morphisms. In this case, the n-simplices are just the set of diagrams

$$* \xrightarrow{g_1} * \xrightarrow{g_2} \cdots \xrightarrow{g_n} *$$

2 b) Describe the non-degenerate simplices of $BG_\bullet$.

2 c) Recall that when $X_\bullet$ is a simplicial set, then $C_*(X_\bullet)$ is the chain complex which in degree n is quotient group of the free Abelian group generated by the set X_n, modulo the free Abelian group generated by all the degenerate n-simplices.

Thus, $C_n(X_\bullet)$ is isomorphic to the free Abelian group generated by all the non-degenerate n-simplices of $X_\bullet$. The differential $\hat{d}_n : C_n(X_\bullet) \to C_{n-1}(X_\bullet)$ was defined on an element x as' the alternating sum

$$\hat{d}_n(x) = \sum_{i=0}^{n} (-1)^i d_i(x)\,.$$

The homology of the simplicial set $BG_\bullet$ is called *the group homology of G.*

Let $G = \mathbb{Z}/2$ be the cyclic group of order two. We write $\mathbb{Z}/2$ multiplicatively as $\{1, T\}$ with $T^2 = 1$.

Write down the chain complex $C_*(B\mathbb{Z}/2)$ and calculate the group homology of $\mathbb{Z}/2$.

2 d) Show that $BG_\bullet$ is a Kan complex when G is Abelian.