

Homologie et Cohomologie Exercises 1

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1 Cat

Let $\underline{\text{Cat}}$ be the category of small categories: the objects are small categories and the morphisms are functors between them. Verify the category axioms.

2 Constructions on categories

Let \underline{C} and \underline{D} be two small categories. Verify the category axioms for the following categories:

1. The **arrow category** $\underline{C}^{\rightarrow}$: $Ob \underline{C}^{\rightarrow} = Mor \underline{C}$ and morphisms are pairs (a, b) such that $\forall f \in \underline{C}(A, B), \forall g \in \underline{C}(C, D)$,
 $\underline{C}^{\rightarrow}(f, g) = \{(a, b) \in (\underline{C}(A, C), \underline{C}(B, D)) : bf = ga\}$, i.e the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a \downarrow & & \downarrow b \\ C & \xrightarrow{g} & D \end{array}$$

2. For a fixed $A \in \underline{C}$, the **slice category** \underline{C}/A : $Ob \underline{C}/A$ are pairs $(X, f) \in (Ob \underline{C}, \underline{C}(X, A))$ and morphisms $a : (X \xrightarrow{f} A) \rightarrow (Y \xrightarrow{g} A)$ such that $\forall X, Y \in Ob \underline{C}, \forall f \in \underline{C}(X, A), \forall g \in \underline{C}(Y, A)$,
 $\underline{C}/A((X, f), (Y, g)) = \{a \in \underline{C}(X, Y) : f = ga\}$, i.e the following triangle commutes:

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ & \searrow f & \swarrow g \\ & & A \end{array}$$

What do you think is the category $A \setminus \underline{C}$?

3. The **product category** $\underline{C} \times \underline{D}$ has objects (A, B) for $A \in Ob \underline{C}, B \in Ob \underline{D}$ and morphisms (f, g) for $f \in Morph \underline{C}$ and $g \in Morph \underline{D}$.

4. The **functor category** $\underline{C}^{\underline{D}}$ has objects the functors $F : \underline{D} \rightarrow \underline{C}$ and morphisms the natural transformations between them.

3 Group homomorphisms

Let $\varphi, \psi : G \rightarrow H$ be two group homomorphisms. Let $\Phi, \Psi : \underline{G} \rightarrow \underline{H}$ be the corresponding functors.

When does there exist a natural transformation $\Phi \rightarrow \Psi$?

4 Pull-back and push-out

Let $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be two morphisms in a category \underline{C} with common domain. The **pull-back** of these two morphisms, if it exists, is an object $A \times_C B$ of \underline{C} and morphisms $A \times_C B \xrightarrow{p_a} A$ and $A \times_C B \xrightarrow{p_b} B$ such that the following square commutes:

$$\begin{array}{ccc} A \times_C B & \xrightarrow{p_b} & B \\ p_a \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

Moreover, the pull-back has the following universal property: whenever another object W in \underline{C} , with morphisms $W \xrightarrow{w_a} A$ and $W \xrightarrow{w_b} B$, makes the same diagram commute, then there exists a unique morphism u , such that the following diagram commutes:

$$\begin{array}{ccccc} W & & & & \\ & \searrow \exists! u & & \searrow w_b & \\ & & A \times_C B & \xrightarrow{p_b} & B \\ & \searrow w_a & p_a \downarrow & & \downarrow g \\ & & A & \xrightarrow{f} & C \end{array}$$

1. Prove that the pull-back is unique up to unique isomorphism.
2. Give an explicit expression for the pull-back in the category of sets.
3. What is the pull-back in the category of sets if C is the one point set?
4. The dual notion of pull-back is **push-out**. Give the definition and draw the corresponding diagrams.
5. Give an explicit expression for the push-out in the category of sets.
6. What is the push-out in the category of sets if C is the empty set?

7. Which is the connection between the theorem of Seifert-van Kampen and pull-backs/push-outs?

5 A further definition of a category

Prove that the following definition of a category is equivalent to the definition in the lecture.

$\underline{C} = (C_0, C_1, id, s, t, m)$ with C_0, C_1 classes.
 $id : C_0 \rightarrow C_1, s : C_1 \rightarrow C_0, t : C_1 \rightarrow C_0, m : C_1 \times_{C_0} C_1 \rightarrow C_1$ where
 $C_1 \times_{C_0} C_1 := \{(f, g) \in C_1 \times C_1 : t(f) = s(g)\}$, such that:

1. $s \circ id = Id_{C_0} = t \circ id$
- 2.

$$\begin{array}{ccccc}
 C_1 & \xrightarrow{j_1} & C_1 \times_{C_0} C_1 & \xleftarrow{j_2} & C_1 \\
 & \searrow & \downarrow m & \swarrow & \\
 & Id_{C_1} & C_1 & Id_{C_1} &
 \end{array}$$

with $j_1(f) := (f, Id_{t(f)})$ and $j_2(g) := (Id_{s(g)}, g)$.

3. m is associative:

$$\begin{array}{ccc}
 C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{Id_{C_1} \times m} & C_1 \times_{C_0} C_1 \\
 \downarrow m \times Id_{C_1} & & \downarrow m \\
 C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1
 \end{array}$$

commutes.