Homologie et Cohomologie: May 25, 2007, Exercises 10

1 Properties of the natural transformation α_G

1. Let G be any abelian group. Show that if $0 \to C \to C' \to C''$ is a short exact sequence of chain complexes, then

$$\begin{array}{ccc} H_nC'' \otimes G & \xrightarrow{\alpha_G(C'')} & H_n(C'' \otimes G) \\ {\scriptstyle \partial \otimes Id_G}\downarrow & & \downarrow{\scriptstyle \partial} \\ H_{n-1}C \otimes G & \xrightarrow{\alpha_G(C)} & H_{n-1}(C \otimes G) \end{array}$$

   commutes, where ∂ is the obvious connecting homomorphism.

2. Let C be any chain complex. Show that for all $\varphi \in \mathbf{Ab}(G, G')$, the diagram

$$\begin{array}{ccc} H_nC \otimes G & \xrightarrow{\alpha_G(C)} & H_n(C \otimes G) \\ {\scriptstyle Id_{H_nC} \otimes \varphi}\downarrow & & \downarrow{\scriptstyle H_n(Id_C \otimes \varphi)} \\ H_nC \otimes G' & \xrightarrow{\alpha_{G'}(C)} & H_n(C \otimes G') \end{array}$$

   commutes.

2 Properties of Tor

Let A, A', B and B' be abelian groups.

1. Show that the definition of $\mathrm{Tor}(A, B)$ is independent of the free resolution of A chosen.

2. Show that $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.

3. Show that $\mathrm{Tor}(A, B \oplus B') \cong \mathrm{Tor}(A, B) \oplus \mathrm{Tor}(A, B')$.

4. Let $\varphi \in \mathbf{Ab}(A, A')$, $\psi \in \mathbf{Ab}(B, B')$. Show that φ induces a homomorphism

$$\mathrm{Tor}(\varphi, \psi) : \mathrm{Tor}(A, B) \to \mathrm{Tor}(A', B')$$

   compatible with the connecting homomorphism in the long exact sequence for $\otimes$ and Tor.

5. Using the construction of the previous point, show that $\mathrm{Tor}(-, B) : \mathbf{Ab} \to \mathbf{Ab}$ is a functor.

3 Calculations of Tor

1. $\operatorname{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) =?$
2. $\operatorname{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}) =?$
3. $\operatorname{Tor}(\mathbb{Z}/m\mathbb{Z}, A) =?$ (A an arbitrary abelian group)

Observe that Exercises 2.2, 2.3 and 3.1, together with the calculation $\operatorname{Tor}(A, \mathbb{Z}) = 0$ done in class, permit us to give a general formula for $\operatorname{Tor}(A, B)$, when A and B are finitely generated abelian groups.

4 The Bockstein homomorphism

Let $C = (C_*, d_*)$ be a chain complex of *free* abelian groups. Let p be any prime. Prove that there is a long exact sequence

$$\cdots \to H_n(C; \mathbb{Z}/p\mathbb{Z}) \to H_n(C; \mathbb{Z}/p^2\mathbb{Z}) \to H_n(C; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\beta_n} H_{n-1}(C; \mathbb{Z}/p\mathbb{Z}) \to \cdots .$$

The connecting homomorphism β_n is called the *mod p Bockstein homomorphism.* What does it mean if β_n is zero? If it is a surjection?