1 Equivalent relations

Show that chain homotopies give rise to an equivalence relation in $\text{Ch}((C_*, d), (C'_*, d'))$ $\forall (C_*, d), (C'_*, d') \in \text{ObCh}$.

2 (Non-)homotopic chain complexes

Give examples of morphisms of chain complexes that are (or are not) homotopic.

3 Isomorphisms in homology and chain homotopies

Give an example of two chain complexes with isomorphic homology groups but not of the same homotopy type.

4 Long exact sequence

Complete the proof that there exists a long exact sequence in homology coming from a short exact sequence of chain complexes.

5 Homomorphisms of a long exact sequence

Given the following chain complexes ($p \in \mathbb{N}$):

- $(C_*, d)$ with
  
  \[ C_n = \begin{cases} 
  \mathbb{Z} & \text{for } n \text{ even} \\
  0 & \text{for } n \text{ odd}
  \end{cases} \]

- $(C'_*, d')$ with $C'_n = \mathbb{Z}$ $\forall n$ and $d'_{2n+1} = \text{multiplication with } p$, $d'_{2n} = 0$ $\forall n$.

- $(C''_*, d'')$ with
  
  \[ C''_n = \begin{cases} 
  \mathbb{Z} & \text{for } n \text{ odd} \\
  0 & \text{for } n \text{ even}
  \end{cases} \]
1. Show that there exists an exact sequence

\[ 0 \longrightarrow (C_\ast, d) \longrightarrow (C'_\ast, d') \longrightarrow (C''_\ast, d'') \longrightarrow 0. \]

2. Write down explicitly all the homomorphisms of the induced long exact sequence.

6 A further chain complex construction

Let \((C_\ast, d), (C'_\ast, d')\) be two chain complexes and let be given a family of maps \(\{\theta_n : C'_n \rightarrow C_{n-1}, \ n \in \mathbb{Z}\}\) such that \(d_n \theta_{n+1} + \theta_n d'_{n+1} = 0\).

Define a third chain complex \((C''_\ast, d''):\)
\[ C''_n = C_n \oplus C'_n \text{ with } d''_n c = d_n c \text{ and } d''_n c' = d'_n c' + \theta_n c'. \]

1. Show that \((C''_\ast, d'')\) is a chain complex.

2. Show that the short exact sequence

\[ 0 \longrightarrow C \longrightarrow C'' \longrightarrow C' \longrightarrow 0 \]

is exact.