Homologie et Cohomologie, May 18, 2007, Exercises 9

1 Some tensor product calculations

1. Show that $M \otimes_R R = M$ for any R module M.
2. $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} \cong ?$
3. $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong ?$
4. Show that $M \otimes_R (\bigoplus_{i \in I} N_i) \cong \bigoplus_{i \in I}(M \otimes_R N_i)$ for M a right R-module and N_i left R-modules $\forall i \in I$.

2 Exact sequences

A functor $F : R \to S$ is called **left exact**, if the exactness of the sequence $0 \to A \to B \to C$ implies exactness of the sequence $0 \to F(A) \to F(B) \to F(C)$,
F is called **right exact**, if the exactness of the sequence $A \to B \to C \to 0$ implies exactness of the sequence $F(A) \to F(B) \to F(C) \to 0$ for $A, B, C \in Ob(R)$.

1. Give examples to show that $- \otimes B$ is not left exact and that $Hom(B, -)$ is not right exact.
2. Prove that if
$$0 \longrightarrow C \longrightarrow C' \longrightarrow C'' \longrightarrow 0$$
is split exact, then
$$0 \longrightarrow C \otimes_{\mathbb{Z}} A \longrightarrow C' \otimes_{\mathbb{Z}} A \longrightarrow C'' \otimes_{\mathbb{Z}} A \longrightarrow 0$$
is also split exact.
3. Show that if C'' is free abelian, then
$$0 \longrightarrow C \longrightarrow C' \longrightarrow C'' \longrightarrow 0$$
is split exact.

3 Homology and tensor products

Give examples of chain complexes (C_*, d) and groups G which yield different homology groups $H_*(C_*, d)$ and $H_*(C_* \otimes_{\mathbb{Z}} G, d \otimes_{\mathbb{Z}} id_G)$.

4 An adjunction

Let R a ring, A a right R-module, B a left R-module and C an abelian group. Show that there is a natural isomorphism

$$\underline{Ab}(A \otimes_{\mathbb{Z}} B, C) \cong \underline{Ab}(A, Hom_{\mathbb{Z}}(B, C)).$$

5 The tensor functor

Show that $- \otimes_{\mathbb{Z}} -\colon \underline{Ab} \times \underline{Ab} \longrightarrow \underline{Ab}$ **is** a functor.