Homologie et Cohomologie, June 15, 2007, Exercises 12

1 A useful lemma

Let

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$$

be an exact sequence of homomorphisms of abelian groups. Show that there exists the following short exact sequence:

$$0 \longrightarrow coker(f) \longrightarrow C \longrightarrow ker(k) \longrightarrow 0.$$

2 Applications of the Künneth-Formula

Using the Künneth-Formula, calculate

1. $H_*(\mathbb{R}P^2 \times \mathbb{R}P^2, \mathbb{Z})$,
2. $H_*(S^1 \times S^1 \times S^1, \mathbb{Z})$
3. $H_*(T^n, \mathbb{Z})$, where $T^n = S^1 \times \cdots \times S^1$, n factors.

3 The Künneth-Formula and the universal coefficient theorem

Explain how to prove the universal coefficient theorem as a corollary of the Kunneth-Formula.

4 Homology of chain coalgebras

Calculate the homology groups of the following chain coalgebras:

1. $(\mathbb{Z}[1] \oplus \mathbb{Z}[x_n], d, \Delta)$ with $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $dx = 0$.
2. $(\mathbb{Z}[1] \oplus \mathbb{Z}[x_1] \oplus \mathbb{Z}[y_2], d, \Delta)$ with

$$\begin{aligned} dx &= 0 & \Delta(x) &= x \otimes 1 + 1 \otimes x \\ dy &= x & \Delta(y) &= y \otimes 1 + 1 \otimes y + x \otimes x. \end{aligned}$$

3. $(\mathbb{Z}[1] \oplus \mathbb{Z}[x_n, y_n] \oplus \mathbb{Z}[z_{2n}], d, \Delta)$ with

$$\begin{aligned} dx &= 0 & \Delta(x) &= x \otimes 1 + 1 \otimes x \\ dy &= 0 & \Delta(y) &= y \otimes 1 + 1 \otimes y \\ dz &= 0 & \Delta(z) &= z \otimes 1 + 1 \otimes z + x \otimes y. \end{aligned}$$

4. $(\mathbb{Z}[1] \oplus \mathbb{Z}[x_3] \oplus \mathbb{Z}[y_4] \oplus \mathbb{Z}[z_6] \oplus \mathbb{Z}[w_7], d, \Delta)$ with

$$\begin{aligned} dx &= 0 & \Delta(x) &= x \otimes 1 + 1 \otimes x \\ dy &= x & \Delta(y) &= y \otimes 1 + 1 \otimes y \\ dz &= 0 & \Delta(z) &= z \otimes 1 + 1 \otimes z + x \otimes x \\ dw &= z & \Delta(w) &= w \otimes 1 + 1 \otimes w + y \otimes x. \end{aligned}$$

5 Comultiplication in homology

As graded abelian groups, $H_*(S^1 \times S^1) \cong H_*(S^1 \vee S^1 \vee S^2)$. That means that the homology groups can not distinguish between $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$. But: the comultiplications in $H_*(S^1 \times S^1)$ and $H_*(S^1 \vee S^1 \vee S^2)$ are distinct! Show this.
Recall that the comultiplication in homology (if there is no torsion in $H_*(X)$) is induced by the diagonal Δ and the Alexander-Whitney chain map

$$C_*(X) \xrightarrow{C_*\Delta} C_*(X \times X) \xrightarrow{f} C_*(X) \otimes C_*(X).$$

The Alexander-Whitney chain map f

$$C_*(K_\bullet \times L_\bullet) \xrightarrow{f} C_*(K_\bullet) \otimes C_*(L_\bullet)$$

is given by

$$f(x, y) = \sum_{i=0}^{n} \bar{d}^{n-i}(x) \otimes d_0^i(y), \qquad x \in K_n, y \in L_n$$

where $\bar{d}(x) = d_n(x)$ and $\bar{d}^{n-i}(x) = d_{i+1}...d_n(x)$.

6 The Alexander-Whitney map

Check that the Alexander-Whitney map $f_{K,L} : C_*(K \times L) \to C_*(K) \otimes C_*(L)$ is indeed a map of chain complexes and that $\nabla_{K,L} \circ f_{K,L} = id$.